Comprehensive Exam – Analysis (January 2010)
Answer 4 out of 5 questions

1. Let $(S,\, d)$ be a metric space.
(a) Give the definition of a Cauchy sequence $\{x_n\}$ in $(S,\, d)$.

(b) Prove that every convergent sequence in $(S,\, d)$ is a Cauchy sequence.

(c) Give an example of a metric space which has a Cauchy sequence that does not converge.

2. Let $f_n(x) = x^n(1 - x^n),\ x \in [0,\, 1]$.
(a) Show that $f_n \to 0$ point-wise, for all $x \in [0,\, 1]$. Justify your answer.

(b) Find the maximum of $f_n(x)$ on $[0,\, 1]$. (Hint: take $u = x^n$).

(c) Using your result of part (b) or otherwise, show that the sequence $\{f_n\}$ does *not* converge uniformly on $[0,\, 1]$.

3. (a) Determine if each of the following series converge. Justify your answer.

$$(i)\ \sum_{n=0}^{\infty} \sqrt{n+1} - \sqrt{n} \qquad (ii)\ \sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2} \qquad (iii)\ \sum_{n=1}^{\infty} \frac{n^n}{n!}\,.$$

(b) Find the interval of convergence and the sum of the power series $1 + x + x^2 + \cdots$.

(c) Use the result of part (a) to compute the sum $\sum_{n=1}^{\infty} nx^n$. State any theorems you need to use to justify your computation.

4. (a) Let $\{a_n(x)\}$ be a sequence of functions defined on an interval $I \subseteq \mathbb{R}$. Then state and prove the Weierstrass M-test for uniform convergence of the series $\sum_{n=0}^{\infty} a_n(x)$ on I.

(b) Using the Weierstrass M-test or otherwise, prove that the series $\sum_{n=0}^{\infty} \frac{\ln(nx)}{n^2}$ represents a continuous function on the interval $[1,\, 4]$.

5. (a) Give an example of a sequence of functions $f_n : [0,1] \to \mathbb{R}$ such that $f_n \to 0$ point-wise on $[0,\, 1]$ but $\int_0^1 f_n \nrightarrow 0$.

(b) Let $f_n : [a,\, b] \to \mathbb{R}$ be a sequence of continuous functions such that $f_n \to f$ on $C([a,\, b], \mathbb{R})$. Then prove that $\int_a^b f_n \to \int_a^b f$.