

**Comprehensive Exam – Linear Algebra**  
**Spring 2007**

1. Determine if each of the following statements is TRUE or FALSE by giving a short proof or a counterexample.

- (a) If  $A$  is a nonsingular square matrix which is diagonalizable, then so is  $A^{-1}$ .
- (b) If a linear operator  $T$  on a *real* inner product space  $V$  satisfies  $\langle Tv, v \rangle = 0$  for all  $v \in V$ , then  $T$  must be the zero transformation.

(c) The matrices  $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$  and  $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$  are similar.

2. For a fixed  $a \in \mathbb{R}$ , consider the subspace  $W = \{f \in P_n(\mathbb{R}) \mid f(a) = f'(a) = 0\}$  of  $P_n(\mathbb{R})$ , the space of real polynomials with degree at most  $n$ . Determine the dimension of  $W$  and write a basis for  $W$ .

3. Given a linear operator  $T$  on a finite dimensional vector space  $V$ , satisfying  $T^2 = T$ .

- (a) Using the dimension theorem, show that  $N(T) \oplus R(T) = V$ .
- (b) Identify all eigenvalues of  $T$  and the corresponding eigenspaces and show that  $T$  is diagonalizable.

4. (i) If  $A$  and  $Q$  are unitary matrices, show that  $U = Q^{-1}AQ$  is also unitary.

(ii) If a unitary matrix  $U$  is also upper triangular, show that  $U$  must be diagonal.

(iii) State Schur's theorem and apply it together with parts (i) and (ii) to show directly that any unitary operator  $T$  on a finite dimensional *complex* inner product space is diagonalizable.

(iv) Does the conclusion in part (iii) remain valid if  $V$  is a *real* inner product space? Justify your answer.

5. Given the  $4 \times 4$  matrix  $A = \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ ,

(a) Find the Jordan canonical form  $J$  of the matrix  $A$  [You do NOT need to compute generalized eigenvectors in this part].

(b) Determine the minimal polynomial of  $A$ .

(c) Show that  $A$  is nonsingular and express  $A^{-1}$  as a polynomial (of least degree) of  $A$ .

(d) Find an invertible matrix  $Q$  such that  $A = QJQ^{-1}$ .