Spring 2003

Comprehensive Exam in Analysis, January 2003

1. (a) State the Monotone Convergence Theorem for sequences of real numbers $\{a_n, n \geq 1\}$.
(b) Let $a_1 = \frac{1}{\sqrt{1}}\frac{1}{2}$, $a_2 = \frac{1}{\sqrt{1}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{1}{2^2}$, $a_3 = \frac{1}{\sqrt{1}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{1}{2^2} + \frac{1}{\sqrt{3}}\frac{1}{2^3}$, ...,

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}\frac{1}{2^k}$$

Prove that $\lim a_n$ exists.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous at $x = 0$ and assume $f(0) = 2$.
(a) State the "ε, δ" definition of the continuity property.
(b) Show there is some $r > 0$ such that $f(x) \geq 1$ for all $x \in (-r, r)$.

3. Let $f(x)$ be monotone increasing on $[0, 1]$, and denote by $L(f, P)$ and $U(f, P)$ the lower and upper sums of f with respect to a partition P. Let $\varepsilon > 0$. Show directly that there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$.

4. Show that the infinite series of functions

$$\sum_{n=1}^{\infty} \left(\frac{x^{2n}}{1 + x^{2n}} \right) \frac{1}{n^2}$$

converges to a continuous function on $\mathbf{R}$. What property of the convergence is being used to obtain the continuity of the infinite sum?

5. (a) State the Contraction Mapping theorem for a complete metric space X.
(b) Use the Mean Value theorem to show that if $f : \mathbf{R} \to \mathbf{R}$ has a derivative satisfying $|f'(x)| \leq \lambda$ for all $x \in \mathbf{R}$ with some constant $\lambda < 1$, then f is a contraction on $\mathbf{R}$.
(c) Apply the Contraction Mapping theorem to establish that there is exactly one soultion to the equation

$$\sin(\tfrac{1}{2}\cos(x)) = x, \quad x \in \mathbf{R}$$

6. Let X be a metric space with metric d. (a) Define what it means for a set A in X to be "open".
(b) Let $x_0 \in X$. Establish by this definition that the neighborhood $A := \{x \in X : d(x_0, x) < 1\}$ is indeed "open".

7. Let $f : \mathbf{R}^2 \to \mathbf{R}$ have continuous partial derivatives on $\mathbf{R}^2$ and let $g : \mathbf{R} \to \mathbf{R}$ be continuously differentiable on $\mathbf{R}$. Show that $g \circ f$ also has continuous partial derivatives on $\mathbf{R}^2$.

8. Suppose that $\{x_n\}$ is a Cauchy sequence in a compact metric space K. Show directly from the definitions of "Cauchy sequence" and "compact set" that the sequence converges in K.