Department of Mathematics

Analysis Comprehensive Examination - Spring 2001

1. Let (X, d) be a metric space and let a be a point in X. Define the function $f(x) = d(a, x)$ for x in X.

(a) Show that f is a continuous function.

(b) Assume that the sequence x_n converges to some x_0 in (X, d). What is the limit of the sequence $d(a, x_n)$? Justify your answer.

2. Let $f_n(x) = e^{-nx}$ for $n \geq 1$ and $x \geq 0$.

(a) Show that f_n has a pointwise limit on $[0, \infty)$.

(b) Does f_n converge uniformly? Support your claim with a rigorous argument.

3. Let $f : R^2 \to R$ be such that $f(tx, ty) = t^3 f(x, y)$ for any (x, y) in R^2 and any t in R.

(a) Give an example of such a function.

(b) Assume that f is differentiable. Show that

$$x \frac{\partial f}{\partial x}(x, y) + y \frac{\partial f}{\partial y}(x, y) = 3f(x, y).$$

4. (a) Define what it means for a real valued function f to be uniformly continuous on $(0, 1]$.

(b) Assuming that f is a real valued function uniformly continuous on $(0, 1]$, show that f is bounded on $[0, 1)$. (Hint: you may want to proceed in 3 steps. First show that there is a $\delta > 0$ such that f is bounded on $[0, \delta)$, second you know that f is bounded on $[\delta, 1]$ (why?), third use these two bounds to show that f is bounded on $(0, 1]$.

(c) If we assume that the function f is only continuous (instead of uniformly continuous) on $(0, 1]$, is it still true that f is bounded? Prove this or give a counter-example.

5. a) Determine whether the following series converge or diverge. Justify your answer.

$$\sum_{k=1}^{\infty} \frac{1}{\pi^k}, \quad \sum_{k=1}^{\infty} \frac{1}{2k+1}.$$

b) Prove that the series

$$\sum_{k=0}^{\infty} x^k / k!$$

converges uniformly on $[-a, a]$ for any real constant a.

6. Suppose that $f : [0, 1] \to R$ is continuous and nonnegative. Prove that if

$$\int_0^1 f(x)\, dx = 0,$$

then $f(x) = 0$ for all $x \in [0, 1]$.

Spring 2001

Comprehensive Exam in Analysis, April 2001

1.(a) Give an example of a metric space which is not complete. Justify your answer.

(b) Prove that a subset K of a complete metric space X is complete (with the same metric) if and only if K is closed.

2. Prove that

$$f(x) = \sum_{j=1}^{\infty} \frac{\cos(4^j x)}{4^j}$$

is continuous everywhere on the real line.

3. Starting with the formula

$$\sum_{j=1}^{n} x^j = \frac{1 - x^{n+1}}{1 - x}$$

find a power series expansion for $\ln(1+x)$. Justify each step.

4. (a) State a theorem that relates connectedness to continuity.

(b) Let A be a connected subset of R^3 such that (1,0,1) and (-1,1,2) belong to A. Explain why there must be in A a point whose first coordinate is 0.

5. (a) Let g be a differentiable function from R^2 to R. Show that if there is a local maximum at (0,0) then we must have $\frac{\partial g}{\partial x}(0,0) = \frac{\partial g}{\partial y}(0,0) = 0$.

(b) If $\frac{\partial g}{\partial x}(0,0) = \frac{\partial g}{\partial y}(0,0) = 0$ does this imply that there is a local extremum at (0,0)? Prove it or give a counterexample.

6. Let f be a continuous function on [0,1] and let x_n be a sequence in [0,1]. Assume that $f(x_n)$ converges to some y_0. Show that there is x_0 in [0,1] such that $f(x_0) = y_0$.