Spring 2000

Comprehensive Exam - Linear Algebra

1. Let P_3 be the vector space of polynomials in x of degree 3 or less with real coefficients. Consider the linear transformation $T : P_3 \to P_3$ defined as $T[p(x)] := \frac{d^2}{dx^2} p(x)$, $p(x) \in P_3$.

6 (a) Show that the null space (kernel) N_T and the range R_T of the linear transformation T satisfy $N_T = R_T$ by finding a basis for each of these subspaces.

8 (b) Let E and O be the sets of even and odd polynomials respectively, that is, $e(-x) = e(x)$, $e(x) \in E$ and $o(-x) = -o(x)$, $o(x) \in O$. Prove the following.

2 (i) E and O are subspaces of P_3.

3 (ii) $T[e(x)] \in E$ and $T[o(x)] \in O$, for every $e(x) \in E$ and $o(x) \in O$.

3 (iii) Every vector $p(x) \in P_3$ can be expressed as $p(x) = e(x) + o(x)$ where $e(x) \in E, o(x) \in O$, and $E \bigcap O = \{0\}$ – the zero vector of P_3.

6 (c) Find a basis of the form $\{e(x), T[e(x)]\}$ for E and of the form $\{o(x), T[o(x)]\}$ for O. Then write down the matrix M that represents T with respect to the ordered basis $B = \{T[e(x)], e(x), T[o(x)], o(x)\}$. What is M^2?

7+8=15 **2.** (a) Suppose an n $\times$ n matrix A has *distinct* eigenvalues. Show that there exists an n $\times$ n matrix B such that $B^2 = A$.

(b) Find a 2×2 matrix B such that

$$B^2 = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

20 **3.** Suppose V is an inner product space over C (the field of complex numbers), and $U : V \to V$ is a linear transformation satisfying $<U\mathbf{v}, U\mathbf{w}>=<\mathbf{v}, \mathbf{w}>$ for vectors $\mathbf{v}, \mathbf{w} \in V$.

5 (a) Let $||\mathbf{v}|| :=<\mathbf{v}, \mathbf{v}>^{1/2}$ be the vector norm induced by this inner product. Show that $||U\mathbf{v}|| = ||\mathbf{v}||$ for every $\mathbf{v} \in V$.

5 (b) Prove that the linear transformation U is non-singular.

5 (c) Let λ be an eigenvalue of U. Show that $\lambda\bar{\lambda} = 1$ where $\bar{\lambda}$ is the complex conjugate of λ.

5 (d) Let M be the matrix representation of U with respect to any *orthonormal* basis B of V. Show that M is a unitary matrix.

4+6=10 **4.** (a) If a matrix A is *both* hermitian and unitary, then show that each eigenvalue of A is either 1 or -1.

(b) If H is a hermitian matrix then prove that the matrices $I \pm iH$ are *nonsingular* and the matrix $U = (I - iH)(I + iH)^{-1}$ is unitary.

15 5. Consider a 5×5 matrix A with minimum polynomial $m(\lambda) = (\lambda+1)^2(\lambda-1)$.

3 (a) List all possible characteristic polynomials for A.

6 (b) Calculate the determinant and trace of A^{-1} for each case listed in part (a).

6 (c) List all possible inequivalent (that is, not similar) Jordan canonical forms J such that $A = PJP^{-1}$.

20 6. Given the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

10 (a) Determine the Jordan canonical form J and the matrix P such that $A = PJP^{-1}$

4 (b) Find the minimal polynomial of A.

6 (c) Compute J^3 and e^J.