Real Analysis Comprehensive Exam
Spring 1998

Do only 3 of the 4 problems on this page

1. Assume f is a continuous function on $(0, 1]$. Does f need to be bounded on (0,1]? Prove it or give a counter-example.

2. Let $f_n(x) = e^{-nx}$.
(a) Show that f_n converges pointwise on $[0, \infty)$.
(b) Does f_n converge uniformly on $[0, \infty)$?

3. Let K and L be compact subsets of R. Show that $K \times L$ is a compact subset of R^2. Use the definition of compactness to prove the claim.

4. Assume that f is **uniformly** continuous on (0,1]. The point of this problem is to show that f is bounded on (0,1]. You may either prove this directly or follow the steps (a), (b) and (c) below.
(a) Show that there is $\delta > 0$ such that f is bounded on $(0, \delta)$. Hint: you may show that there is $\delta > 0$ such that for all $x \in (0, \delta)$ we have $|f(x)| \leq 1 + |f(\delta)|$.
(b) Show that f is bounded on $[\delta, 1]$.
(c) Show that f is bounded on (0,1).

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5. Define a sequence of functions on the line by

$$f_n(x) = \frac{n}{1+n^2x^2}$$

(a) Show that the sequence converges a.e. wrt Lebesgue measure m and find the limit function f.

(b) Show that f_n does not converge in $L^1(R, \mathcal{L}, m)$.

(c) Find an interval I of R for which f_n converges on $L^1(I, \mathcal{L}, m)$.

6. Give an outline of the major features of the development of the Lebesgue measure and integration on the line. If more familiar, you may use the context of Lebesgue-Stieltjes measures and integrals definided by monotonically increasing functions, but if you do so point out the special features of Lebesgue measure and integration. It is also OK to stick to the Lebesgue case only.

7. Define the Cantor set on [0,1].

(a) Show that the Cantor set has measure 0.

(b) Show that the Cantor set is uncountable.

8. Define the following function on R^2

$$f(x,y) = \frac{\sin(xy)}{1+|x|+|y|}e^{-(x^2+y^2)}$$

(a) Use the Fubini-Tonelli theorem to show that f is in $L^1(R^2, m \times m)$.

(b) Show that the function

$$g(x,y) = \frac{1}{1+x^2+y^2}$$

is in $L^1(R, m)$ in the variable x for each fixed y, but g is not in $L^1(R^2, m \times m)$.