Distribution of Runs in Gambler's Ruin

Gregory J. Morrow

Abstract

The arcsine law is classical result of probability theory. We toss a fair coin independently $2n$ times. Denote by $2k$ the number of the toss on which the number of heads equals the number of tails for the last time before the $2n^{\text{th}}$ toss. Denote the random variable $X_n = 2k/2n$. The arcsine law states that the limiting random variable $X = \lim X_n$ has a density $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$, $0 < x < 1$, with cumulative distribution function $F(x) = P(X \leq x) = \frac{2}{\pi}\arcsin(\sqrt{x})$, $0 \leq x \leq 1$.

In this talk we first show how to extend the arcsine law to the Gambler's Ruin setting, wherein we continue to toss the coin until the fortune $:=$ #heads $-$ #tails satisfies $|\text{fortune}| = N$ for a large positive integer N. The expected number of tosses in the Gambler's Ruin is N^2. If $\mathbf{T}_N$ denotes the number of the toss of the "last visit" by the fortune to the initial value 0, then $\mathbf{X}_N = \mathbf{T}_N/N^2$ converges in law as $N \to \infty$ to a random variable $\mathbf{X}$ with density $f(x) = (\pi x)^{-1/2}\sum_{\nu=-\infty}^{\infty}(-1)^\nu e^{-\nu^2/x}$, $x > 0$. Secondly we show a path decomposition method that extends from the number of tosses or steps in a random walk determined by the coin tossing, to the number of runs in this random walk. Here a single run consists of a series of tosses on which the coin lands up the same each time. The approach is combinatorial in nature, and one of the facets of the method requires one to develop a closed form of the generating function $\sum_{N=0}^{\infty} F_N(x)y^N$ of a sequence of certain polynomials $\{F_N(x), N \geq 0\}$ with combinatorial formulae involving binomial coefficients. The Fibonacci-Chebyshev polynomials $F_N(x)$ are introduced as the simplest example of such polynomials, associated to the "last visit" in Gambler's Ruin.