On Factorizations of Differential Operators and Hardy-Rellich-Type Inequalities

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Hardy–Rellich-Type Inequalities:

• Derive the **basic inequality**

$$\begin{aligned}\int_{\mathbb{R}^n} |(\Delta f)(x)|^2\, d^n x \geq [(n-4)\alpha - 2\beta] \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2\, d^n x \\ - \alpha(\alpha-4) \int_{\mathbb{R}^n} |x|^{-4} |x \cdot (\nabla f)(x)|^2\, d^n x \\ + \beta[(n-4)(\alpha-2) - \beta] \int_{\mathbb{R}^n} |x|^{-4} |f(x)|^2\, d^n x, \\ \alpha, \beta \in \mathbb{R},\ f \in C_0^\infty(\mathbb{R}^n \backslash \{0\}),\end{aligned}$$

and some **variations** of it.

• Specialize the parameters α, β to arrive at well-known inequalities, such as the **Rellich inequality** and some of its ramifications.

• Use **factorizations** of differential operators ($L = T^* T \geq 0$) as a tool to derive such inequalities.

• Illustrate the great **flexibility** and **simplicity** of this factorization approach.

Motivation and Some Literature:

Motivation: **Hardy-type inequalities** are at the center of certain **self-adjointness** proofs; they are fundamental in proving **lower boundedness** of Hamiltonians, **relative form boundedness,** etc. They're an ubiquitous presence in spectral theory .....

The Emphasis lies on the Method Employed: This is not an attempt to find one more elegant/short proof of Hardy-type inequalities. There exist many such proofs already. Rather, we present an **elementary method**, based on **factorizations** of even-order differential expressions that's **remarkably flexible**: It reproduces the well-known inequalities, but also less well-known ones, and even **new ones**, and in many cases produces **best constants.**

Based on:

F.G. and L. Littlejohn, *Factorizations and Hardy–Rellich-type inequalities*, to appear in *Partial Differential Equations, Mathematical Physics, and Stochastic Analysis. A Volume in Honor of Helge Holden's 60th Birthday*, EMS Congress Reports, arXiv:1701.08929.

F.G., L. Littlejohn, I. Michael, and R. Wellmann, *On Birman's sequence of Hardy–Rellich-type inequalities*, preprint, 2017.

Hardy–Rellich-type Inequalities on $(0,\infty)$:

Consider the differential expressions

$$T = \frac{d}{dx} + \frac{\alpha}{x}, \quad T^+ = -\frac{d}{dx} + \frac{\alpha}{x}, \quad x > 0,$$

with $\alpha, \beta \in \mathbb{R}$ (**homogeneous** of degree -1), which are formal adjoints to each other. Then

$$T^+ T = -\frac{d^2}{dx^2} + \frac{\alpha^2 + \alpha}{x^2},$$

and hence integrating by parts,

$$\begin{aligned} 0 &\leq \int_0^\infty (Tf)(x)^2\, dx = \int_0^\infty f(x)(T^+ Tf)(x)\, dx \\ &= \int_0^\infty [f'(x)]^2\, dx + (\alpha^2 + \alpha) \int_0^\infty \frac{f(x)^2}{x^2}\, dx, \quad f \in C_0^\infty((0,\infty)), \end{aligned}$$

choosing f real-valued w.l.o.g. Thus, one gets the **Hardy-type inequality**

$$\int_0^\infty |f'(x)|^2\, dx \geq -(\alpha^2 + \alpha) \int_0^\infty \frac{|f(x)|^2}{x^2}\, dx,$$
$$\alpha, \beta \in \mathbb{R}, \; f \in C_0^\infty((0,\infty)).$$

Hardy–Rellich-type Inequalities on $(0, \infty)$ (contd.):

Maximizing w.r.t. α yields **Hardy's classical inequality** for the half-line

$$\boxed{\int_0^\infty |f'(x)|^2\, dx \geq \frac{1}{4} \int_0^\infty \frac{|f(x)|^2}{x^2}\, dx,} \quad f \in C_0^\infty((0,\infty)).$$

It is well-known that $1/4$ **is optimal** and **the inequality is strict**, i.e., equality holds if and only if $f \equiv 0$.

Hardy–Rellich-type Inequalities on $(0, \infty)$ (contd.):

Of course, **that's a really old hat!** **Hardy, 1915, 1919, etc.**

But, emboldened by this, we march on: Next, consider

$$T = -\frac{d^2}{dx^2} + \frac{\alpha}{x}\frac{d}{dx} + \frac{\beta}{x^2}, \quad T^+ = -\frac{d^2}{dx^2} - \frac{\alpha}{x}\frac{d}{dx} + \frac{\alpha+\beta}{x^2}, \quad x > 0,$$

with $\alpha, \beta \in \mathbb{R}$ (the differential expressions are **homogeneous** of degree -2), which are formal adjoints to each other. Then,

$$T^+T = \frac{d^4}{dx^4} + \frac{\alpha - \alpha^2 - 2\beta}{x^2}\frac{d^2}{dx^2} + \frac{2\alpha^2 - 2\alpha + 4\beta}{x^3}\frac{d}{dx} + \frac{3\alpha\beta + \beta^2 - 6\beta}{x^4},$$

and upon integrating by parts,

$$\begin{aligned} 0 \leq \int_0^\infty (Tf)(x)^2\,dx &= \int_0^\infty f(x)(T^+Tf)(x)\,dx \\ &= \int_0^\infty [f''(x)]^2\,dx - \big(\alpha - \alpha^2 - 2\beta\big)\int_0^\infty \frac{[f'(x)]^2}{x^2}\,dx \\ &\quad + \beta(3\alpha + \beta - 6)\int_0^\infty \frac{f(x)^2}{x^4}\,dx, \quad f \in C_0^\infty((0,\infty)), \end{aligned}$$

Hardy–Rellich-type Inequalities on $(0, \infty)$ (contd.):

again choosing w.l.o.g. f real-valued. Thus,

$$\int_0^\infty |f''(x)|^2 \, dx \geq \left(\alpha - \alpha^2 - 2\beta\right) \int_0^\infty \frac{|f'(x)|^2}{x^2} \, dx + \beta(6 - \beta - 3\alpha) \int_0^\infty \frac{|f(x)|^2}{x^4} \, dx, \quad f \in C_0^\infty((0,\infty)), \; \alpha, \beta \in \mathbb{R}.$$

Choosing $\beta = \left(\alpha - \alpha^2\right)/2$ yields the **Rellich-type inequality**

$$\int_0^\infty |f''(x)|^2 \, dx \geq \left[3\alpha - (19/4)\alpha^2 + 2\alpha^3 - (1/4)\alpha^4\right] \int_0^\infty \frac{|f(x)|^2}{x^4} \, dx, \quad f \in C_0^\infty((0,\infty)).$$

Hardy–Rellich-type Inequalities on $(0, \infty)$ (contd.):

Maximizing w.r.t. α yields **Rellich's classical inequality** for the half-line

$$\boxed{\int_0^\infty |f''(x)|^2\,dx \geq \frac{9}{16} \int_0^\infty \frac{|f(x)|^2}{x^4}\,dx,} \quad f \in C_0^\infty((0,\infty)).$$

Again, $9/16$ **is optimal** and **the inequality is strict**, i.e., equality holds if and only if $f \equiv 0$.

History: Not entirely clear to us. **Rellich's** book dates from 1969 and treats the mult-dimensional case, but **Birman** had this in 1961 (translated in 1966), however, he provides no references ......

Birman's Sequence of Inequalities on $(0,\infty)$:

Actually, the Rellich inequality is not the end, it's just the beginning: Birman presented in 1961 (almost in passing) the following sequence of inequalities (AMS Transl. (2) **53**, 23–80 (1966)):

Theorem 1.

$$\boxed{\int_0^\infty |f^{(n)}(x)|^2\,dx \geq \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^\infty \frac{|f(x)|^2}{x^{2n}}\,dx,} \quad n\in\mathbb{N},\ f\in C_0^\infty((0,\infty)).$$

An Extension [GLMW17] (apparently, new): The Birman inequalities work with $C_0^\infty((0,\infty))$ replaced by the space,

$$\begin{aligned} H_n([0,\infty)) &= \big\{f:[0,\infty)\to\mathbb{C}\,\big|\, f^{(j)}\in AC_{loc}([0,\infty)); f^{(n)}\in L^2((0,\infty)); \\ &\qquad f^{(j)}(0)=0,\ j=0,\dots,(n-1)\big\} \\ &= \big\{f:(0,\infty)\to\mathbb{C}\,\big|\, f^{(j)}\in AC_{loc}((0,\infty)),\ j=0,\dots,(n-1); \\ &\qquad x^{-n}f,\ f^{(n)}\in L^2((0,\infty))\big\}. \end{aligned}$$

This appears to be a new observation.

Birman's Sequence of Inequalities (contd.):

Note. (i) Equality between the two spaces above requires a bit of work.
(ii) $H_n([0,\infty))$ does **NOT** equal the standard Sobolev space $H_0^{(n)}((0,\infty))$.

Example. $g(x) = \begin{cases} 0, & \text{near } x = 0, \\ x^{(2n-1)/2}/\ln(x), & \text{near } \infty, \end{cases}$ with $g^j \in AC_{loc}([0,\infty))$, $j = 0,\dots,n$, then $g \in H_n([0,\infty))$, but $g^{(k)} \notin L^2((0,\infty))$, $k = 0,\dots,n-1$.

(iii) $H_n([0,\infty))$ is a Hilbert space with scalar product

$$(f,g)_{H_n([0,\infty))} = \int_0^\infty \overline{f^{(n)}(x)} g^{(n)}(x)\, dx.$$

(The boundary conditions $h^{(j)}(0) = 0$, $j = 0,\dots,(n-1)$, render the kernel of d^n/dx^n trivial.)

A further possible Extension: Let $p \in (1,\infty)$, then

$$\boxed{\int_0^\infty |f^{(n)}(x)|^p\, dx \geq \frac{\prod_{k=1}^n (kp-1)^p}{p^{pn}} \int_0^\infty \frac{|f(x)|^p}{x^{pn}}\, dx,} \quad n \in \mathbb{N},\ f \in C_0^\infty((0,\infty)).$$

Birman's Sequence of Inequalities on $(0,b)$, $b<\infty$:

The Finite Interval Case $(0,b)$**,** $b\in(0,\infty)$**:** Everything is **local,** thus, simply replace $(0,\infty)$ everywhere by $(0,b)$, $C_0^\infty((0,\infty))$ by $C_0^\infty((0,b))$, etc.

One interesting difference, though! Equivalence with the **standard Sobolev space** $H_0^{(n)}((0,b))$:

$$\begin{aligned}H_{n,0}([0,b]) &= \big\{f:[0,b]\to\mathbb{C}\,\big|\, f^{(j)}\in AC([0,b]);\, f^{(n)}\in L^2((0,b));\\ &\qquad\qquad \boxed{f^{(j)}(0)=0=f^{(j)}(b),}\, j=0,\dots,(n-1)\big\}\\ &= \big\{f:(0,b]\to\mathbb{C}\,\big|\, f^{(j)}\in AC_{loc}((0,b]),\, f^{(j)}(b)=0,\, j=0,\dots,(n-1);\\ &\qquad\qquad x^{-n}f,\, f^{(n)}\in L^2((0,b))\big\}\\ &= H_0^{(n)}((0,b)), \quad b\in(0,\infty),\end{aligned}$$

as a consequence of the **Friedrichs inequality**,

$$\big\|f^{(j)}\big\|_{L^2((0,b))}\leq C\big\|f^{(n)}\big\|_{L^2((0,b))}, \quad f\in H_0^n((0,b)),\ b\in(0,\infty),$$

with $C=C(j,n,b)\in(0,\infty)$ independent of $f\in H_0^n((0,b))$.

Birman's Sequence of Ineq. on $(0, b)$, $b < \infty$ (cont.):

Theorem 2 [GLMW17].

Let $n \in \mathbb{N}$, $b \in (0, \infty)$. Then the following items (i)–(iv) hold:

(i) For each $n \in \mathbb{N}$,

$$H_n([0, b]) = H_0^n((0, b))$$

as sets. In particular,

$$f \in H_n([0, b]) \text{ implies } f^{(j)} \in L^2((0, b)), \quad j = 0, 1, \dots, n.$$

In addition, the norms in $H_n([0, b])$ and $H_0^n((0, b))$ are **equivalent**.

(ii) The following hold:

(α) Let $f : [0, b] \to \mathbb{C}$, with $f^{(j)} \in AC([0, b])$, $f^{(j)}(0) = 0$, $j = 0, 1, \dots, n-1$, and $f^{(n)} \in L^2((0, b))$. (**No b.c.'s at endpoint b !**) Then,

$$\int_0^b \left|f^{(n)}(x)\right|^2 dx \geq \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^b \frac{|f(x)|^2}{x^{2n}} dx.$$

Birman's Sequence of Ineq. on $(0, b)$, $b < \infty$ (cont.):

Theorem 2 (contd.) [GLMW17].

(ii) (contd.)

(β) If $f : [a, b] \to \mathbb{C}$, with $f^{(j)} \in AC([0, b])$, $f^{(j)}(b) = 0$, $j = 0, 1, \dots, n-1$, and $f^{(n)} \in L^2((0, b))$. Then,

$$\int_0^b \left|f^{(n)}(x)\right|^2 dx \geq \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^b \frac{|f(x)|^2}{(b-x)^{2n}}\, dx.$$

(γ) Introducing the **distance of $x \in (0, b)$ to the boundary** $\{0, b\}$ of $(0, b)$ by $d(x) = \min\{x, |b - x|\}$, $x \in (0, b)$, one has

$$\int_0^b \left|f^{(n)}(x)\right|^2 dx \geq \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^b \frac{|f(x)|^2}{d(x)^{2n}}\, dx, \quad f \in H_0^n((0, b)).$$

In all cases (α)–(γ), if $f \not\equiv 0$, the above inequalities are **strict**.

(iii) The constant $[(2n-1)!!]^2/2^{2n}$ is **sharp**.

The Vector-Valued Case:

Extensions to the **vector-valued** case: Consider a complex, separable Hilbert space $\mathcal{H}$, the inner product in $L^2((a,b);\mathcal{H})$, in obvious notation, then reads

$$(f,g)_{L^2((a,b);\mathcal{H})} = \int_a^b (f(x),g(x))_{\mathcal{H}}\,dx, \quad f,g \in L^2((a,b);\mathcal{H}).$$

In other words, $L^2((a,b);\mathcal{H})$ can be identified with the constant fiber direct integral of Hilbert spaces, $L^2((a,b);\mathcal{H}) \simeq \int_{(a,b)}^{\oplus} \mathcal{H}\,dx$, and similarly one introduces $H_n([0,\infty);\mathcal{H})$.

Theorem 3 [GLMW17].

For $0 \neq f \in H_n([0,\infty);\mathcal{H})$, one has (with $[(2n-1)!!]^2/2^{2n}$ being **sharp**)

$$\int_0^\infty \big\|f^{(n)}(x)\big\|_{\mathcal{H}}^2\,dx > \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^\infty \frac{\|f(x)\|_{\mathcal{H}}^2}{x^{2n}}\,dx, \quad n \in \mathbb{N}.$$

Note. The case $n=1$ played a role in spectral and scattering theory for Schrödinger operators in $\mathbb{R}^d$ (**Agmon, Kuroda**) with $\mathcal{H} = L^2(S^{d-1}; d^{d-1}\omega)$, $d \in \mathbb{N}$, $d \geq 2$.

The Vector-Valued Case (contd.): $b \in (0, \infty)$

Consider the finite interval case $(0, b)$, $b \in (0, \infty)$ and introduce (with $n \in \mathbb{N}$),

$$H_n([0,b];\mathcal{H}) := \big\{ f : [0,b] \to \mathcal{H} \,\big|\, f^{(n)} \in L^2((0,b);\mathcal{H});\ f^{(j)} \in AC([0,b];\mathcal{H}); \boxed{f^{(j)}(0) = 0 = f^{(j)}(b),}\ j = 0, 1, \dots, n-1 \big\},$$

and the standard $\mathcal{H}$-valued Sobolev spaces,

$$H^n((0,b);\mathcal{H}) = \big\{ f : [0,b] \to \mathcal{H} \,\big|\, f^{(j)} \in AC([0,b];\mathcal{H}),\ j = 0, 1, \dots, n-1; f^{(k)} \in L^2((0,b);\mathcal{H}),\ k = 0, 1, \dots, n \big\},$$

$$H_0^n((0,b);\mathcal{H}) = \big\{ f \in H^n((0,b);\mathcal{H}) \,\big|\, \boxed{f^{(j)}(0) = 0 = f^{(j)}(b),}\ j = 0, 1, \dots, n-1 \big\}.$$

Again, the **vector-valued Friedrichs inequality**

$$\|f\|_{L^2((0,b);\mathcal{H})} \leq b \|f'\|_{L^2((0,b);\mathcal{H})}, \quad f \in H_1([0,b];\mathcal{H})$$

yields $H_1([0,b];\mathcal{H}) = H_0^1((0,b);\mathcal{H})$, and upon iteration,

$$H_n([0,b];\mathcal{H}) = H_0^n((0,b);\mathcal{H}), \quad n \in \mathbb{N}.$$

The Vector-Valued Case (contd.): $b \in (0, \infty)$

Theorem 4 [GLMW17].

Let $n \in \mathbb{N}$, $b \in (0, \infty)$. Then

(i) For each $n \in \mathbb{N}$,

$$H_n([0,b];\mathcal{H}) = H_0^n((0,b);\mathcal{H})$$

as sets. In particular,

$$f \in H_n([0,b];\mathcal{H}) \text{ implies } f^{(j)} \in L^2((0,b);\mathcal{H}), \quad j = 0, 1, \dots, n.$$

In addition, the norms in $H_n([0,b];\mathcal{H})$ and $H_0^n((0,b);\mathcal{H})$ are **equivalent**.

(ii) Recalling $d(x) = \min\{x, |b-x|\}$, $x \in (0,b)$, one has

$$\int_0^b \big\|f^{(n)}(x)\big\|_{\mathcal{H}}^2\, dx \geq \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^b \frac{\|f(x)\|_{\mathcal{H}}^2}{d(x)^{2n}}\, dx, \quad f \in H_0^n((0,b)).$$

If $f \not\equiv 0$, the above inequality is **strict**.

(iii) The constant $[(2n-1)!!]^2/2^{2n}$ is **sharp**.

Much more could be done, but **on to multi-dimensions**.

A Fundamental Inequality:

At first we focus on one point singularity, but eventually illustrate how **any finite number**, even **countably infinitely many**, can be handled in applications.

Theorem 5 (G., Littlejohn, 2016).

Let $\alpha, \beta \in \mathbb{R}$, and $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, $n \in \mathbb{N}$, $n \geq 2$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq [(n-4)\alpha - 2\beta] \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 \, d^n x \\ - \alpha(\alpha - 4) \int_{\mathbb{R}^n} |x|^{-4} |x \cdot (\nabla f)(x)|^2 \, d^n x \\ + \beta[(n-4)(\alpha - 2) - \beta] \int_{\mathbb{R}^n} |x|^{-4} |f(x)|^2 \, d^n x. \end{aligned}$$

In addition, if either $\alpha \leq 0$ or $\alpha \geq 4$, then,

$$\begin{aligned} \int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq [\alpha(n - \alpha) - 2\beta] \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 \, d^n x \\ + \beta[(n-4)(\alpha - 2) - \beta] \int_{\mathbb{R}^n} |x|^{-4} |f(x)|^2 \, d^n x. \end{aligned}$$

A Fundamental Inequality (contd.):

Note. By locality, these inequalities naturally extend to the case where $\mathbb{R}^n$ is replaced by an arbitrary open set $\Omega \subset \mathbb{R}^n$ for functions $f \in C_0^\infty(\Omega\backslash\{0\})$ (without changing the constants in these inequalities).

Sketch of Proof of Theorem 5. Consider, with $\alpha, \beta \in \mathbb{R}$,

$$T_{\alpha,\beta} := -\Delta + \alpha|x|^{-2}x\cdot\nabla + \beta|x|^{-2}, \quad x \in \mathbb{R}^n\backslash\{0\},$$

(**homogeneous** of degree -2) and its formal adjoint, denoted by $T^+_{\alpha,\beta}$,

$$T^+_{\alpha,\beta} := -\Delta - \alpha|x|^{-2}x\cdot\nabla + [\beta - \alpha(n-2)]|x|^{-2}, \quad x \in \mathbb{R}^n\backslash\{0\}.$$

Then, for $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$,

$$\begin{aligned}(T^+_{\alpha,\beta}T_{\alpha,\beta}f)(x) = (\Delta^2 f)(x) &+ [(n-4)\alpha - 2\beta]|x|^{-2}(\Delta f)(x)\\ &+ \alpha(4-\alpha)|x|^{-4}\sum_{j,k=1}^{n} x_j x_k f_{x_j,x_k}(x)\\ &+ \big[-(n-3)\alpha^2 + 2(n-2)\alpha + 4\beta\big]|x|^{-4}x\cdot(\nabla f)(x)\\ &+ \big[\beta^2 + 2(n-4)\beta - (n-4)\alpha\beta\big]|x|^{-4}f(x).\end{aligned}$$

A Fundamental Inequality (contd.):

Again, assuming $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ to be real-valued, integrating by parts (observing the support properties of f, which results in vanishing surface terms), results in (**not without some tears involved ......**)

$$
\begin{aligned}
0 \leq \int_{\mathbb{R}^n} [(T_{\alpha,\beta} f)(x)]^2 \, d^n x &= \int_{\mathbb{R}^n} f(x) (T_{\alpha,\beta}^+ T_{\alpha,\beta} f)(x) \, d^n x \\
&= \int_{\mathbb{R}^n} [(\Delta f)(x)]^2 \, d^n x + [(n-4)\alpha - 2\beta] \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x|^{-2} f(x) (\Delta f)(x) \, d^n x \\
&\quad + \alpha(\alpha - 4) \sum_{j,k=1}^n \int_{\mathbb{R}^n} |x|^{-4} f(x) x_j x_k f_{x_j,x_k}(x) \, d^n x \\
&\quad + \big[-(n-3)\alpha^2 + 2(n-2)\alpha + 4\beta \big] \int_{\mathbb{R}^n} |x|^{-4} f(x) [x \cdot (\nabla f)(x)] \, d^n x \\
&\quad + \big[\beta^2 + 2(n-4)\beta - (n-4)\alpha\beta \big] \int_{\mathbb{R}^n} |x|^{-4} f(x)^2 \, d^n x.
\end{aligned}
$$

A Fundamental Inequality (contd.):

To simplify matters we make two observations. First, a standard integration by parts (again observing the support properties of f) yields

$$\int_{\mathbb{R}^n} |x|^{-2} f(x)(\Delta f)(x)\, d^n x = 2 \int_{\mathbb{R}^n} |x|^{-4} f(x)(x \cdot (\nabla f)(x)\, d^n x - \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2\, d^n x.$$

Similarly, one confirms that

$$\sum_{j,k=1}^{n} \int_{\mathbb{R}^n} x_j x_k f(x) f_{x_j,x_k}(x) = -(n-3) \int_{\mathbb{R}^n} |x|^{-4} f(x)[x \cdot (\nabla f)(x)]\, d^n x - \int_{\mathbb{R}^n} |x|^{-4} [x \cdot (\nabla f)(x)]^2\, d^n x.$$

This yields the 1st inequality in the theorem.

A Fundamental Inequality (contd.):

Since by Cauchy's inequality,

$$-\int_{\mathbb{R}^n} |x|^{-4}[x \cdot (\nabla f)(x)]^2 \, d^n x \geq -\int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 \, d^n x,$$

one concluses that as long as $\alpha(\alpha - 4) \geq 0$, that is, as long as either $\alpha \leq 0$ or $\alpha \geq 4$, one arrives at the 2nd inequality in the theorem. $\square$

In principle, a "nice" calculus exercise!

Believe it or not, this is actually useful as we shall see next:

Consequences of the Fundamental Inequality:

Corollary 6 (Rellich's Inequality).

Let $n \in \mathbb{N}$, $n \geq 5$, and $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Then,

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq \left[\frac{n(n-4)}{4}\right]^2 \int_{\mathbb{R}^n} |x|^{-4} |f(x)|^2 \, d^n x.$$

The constant $[n(n-4)/4]^2$ is known to be **optimal**.

Sketch of Proof. Choosing $\beta = \alpha(n-\alpha)/2$ in the 2nd inequality in Theorem 5 results in

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq G_n(\alpha) \int_{\mathbb{R}^n} |x|^{-4} |f(x)|^2 \, d^n x,$$

with

$$G_n(\alpha) = \alpha(n-\alpha)\{(n-4)(\alpha-2) - [\alpha(n-\alpha)/2]\}/2.$$

Maximizing $G_n(\alpha)$ with respect to α yields Rellich's inequality. □

Consequences of the Fundamental Inequ. (contd.):

Corollary 7

Let $n \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Then,

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq \frac{n^2}{4} \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 \, d^n x, \quad n \geq 8,$$

and

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq 4(n-4) \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 \, d^n x, \quad 5 \leq n \leq 7.$$

In addition,

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq \frac{n^2}{4} \int_{\mathbb{R}^n} |x|^{-4} |x \cdot (\nabla f)(x)|^2 \, d^n x, \quad n \geq 2.$$

Note. The constant $4(n-4)$ for $n = 5, 6, 7$ should be $n^2/4$, so that seems to be one mysterious instance where this method may not yield an optimal constant.

Consequences of the Fundamental Inequ. (contd.):

Sketch of Proof of Corollary 7. Choosing $\beta = 0$ in the 2nd inequality in Theorem 5 yields

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq \alpha(n - \alpha) \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 \, d^n x.$$

Maximizing $F_n(\alpha) = \alpha(n - \alpha)$ with respect to α yields a maximum at $\alpha_1 = n/2$, and subjecting it to the constraint $\alpha \geq 4$ proves the 1st inequality of Corollary 7.

Choosing $\alpha = 4$, $\beta = 0$ in the 1st inequality in Theorem 5 yields the 2nd inequality of Corollary 7.

Consequences of the Fundamental Inequ. (contd.):

The choice $\beta = (n-4)(\alpha-2)$ in the 1st inequality in Theorem 5 results in

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq (n-4)(4-\alpha) \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 \, d^n x \\ - \alpha(\alpha-4) \int_{\mathbb{R}^n} |x|^{-4} |x \cdot (\nabla f)(x)|^2 \, d^n x.$$

For $n \geq 2$ and $(4-n) < \alpha < 4$, applying Cauchy's inequality to the 1st term on the right-hand side yields

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq K_n(\alpha) \int_{\mathbb{R}^n} |x|^{-4} |x \cdot (\nabla f)(x)|^2 \, d^n x,$$

where $K_n(\alpha) = -(\alpha+n-4)(\alpha-4)$. Maximizing K_n subject to the constraint $(4-n) < \alpha < 4$ yields the 3rd inequality of Corollary 7. □

Other Consequences: Schmincke's Inequality

Our method recovers (actually, extends) Schmincke's one-parameter family of inequalities from 1972:

Corollary 8 (Schmincke's 1972 Inequality).

Let $n \in \mathbb{N}$, $n \geq 5$, and $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Then,

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq -s \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 \, d^n x + [(n-4)/4]^2 (4s + n^2) \int_{\mathbb{R}^n} |x|^{-4} |f(x)|^2 \, d^n x,$$
$$s \in \big[-2^{-1} n(n-4), \infty \big).$$

Sketch of Proof. Choose $\beta = 2^{-1}(n-4)[\alpha - 2 - 4^{-1}(n-4)]$, and the introduction of the new variable $s = s(\alpha) = \alpha^2 - 4\alpha - 2^{-1}n(n-4)$, in the fundamental two-parameter inequality in Theorem 5. □

Note. Particular choices of s reproduce Rellich's inequality (Corollary 6) and also some of the inequalities in Corollary 7 as special cases.

Back to Hardy's Inequality and Some Refinements:

I first started to look into factorizations well over 30 years ago: Let $n \geq 3$ and consider

$$T_\alpha := \nabla + \alpha|x|^{-2}x, \quad x \in \mathbb{R}^n\backslash\{0\},$$

with formal adjoint

$$T_\alpha^+ = -\mathrm{div}(\,\cdot\,) + \alpha|x|^{-2}x\,\cdot, \quad x \in \mathbb{R}^n\backslash\{0\},$$

such that (e.g., on $C_0^\infty(\mathbb{R}^n\backslash\{0\})$-functions),

$$T_\alpha^+ T_\alpha = -\Delta + \alpha(\alpha + 2 - n)|x|^{-2}.$$

Repeating earlier steps and optimizing w.r.t. α readily yields **Hardy's classical inequality**

$$\boxed{\int_{\mathbb{R}^n} |(\nabla f)(x)|^2\, d^n x \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} |x|^{-2}|f(x)|^2\, d^n x,} \quad f \in C_0^\infty(\mathbb{R}^n\backslash\{0\}),\ n \geq 3.$$

The constant $(n-2)^2/4$ is **optimal**.

Some Refinements Hardy's Inequality:

Similarly, assuming $n \geq 3$ and introducing the refinement **(radial derivative)**,

$$\widetilde{T}_\alpha := \big(|x|^{-1}x\big) \cdot \nabla + \alpha |x|^{-1}, \quad x \in \mathbb{R}^n \backslash \{0\},$$

with formal adjoint,

$$\big(\widetilde{T}_\alpha\big)^+ = -\big(|x|^{-1}x\big) \cdot \nabla + (\alpha - n + 1)|x|^{-1}, \quad x \in \mathbb{R}^n \backslash \{0\},$$

one computes (e.g., on $C_0^\infty(\mathbb{R}^n \backslash \{0\})$-functions),

$$\begin{aligned}\big(\widetilde{T}_\alpha\big)^+ \widetilde{T}_\alpha = -|x|^{-2} \sum_{j,k=1}^{n} x_j x_k \partial_{x_j} \partial_{x_k} - (n-1)|x|^{-2}[x \cdot (\nabla f)(x)] \\ + \alpha(\alpha + 2 - n)|x|^{-2}, \quad x \in \mathbb{R}^n \backslash \{0\}.\end{aligned}$$

Proceeding as before yields

$$\int_{\mathbb{R}^n} \big|\big[|x|^{-1}x \cdot \nabla f\big](x)\big|^2 \, d^n x \geq \alpha[(n-2)-\alpha] \int_{\mathbb{R}^n} |x|^{-2} |f(x)|^2 \, d^n x, \quad f \in C_0^\infty(\mathbb{R}^n \backslash \{0\}).$$

Some Refinements Hardy's Inequality (contd.):

Maximizing $\alpha[(n-2)-\alpha]$ with respect to α yields the **improved/refined Hardy inequality** involving the **radial derivative**, $\partial/\partial r = |x|^{-1} x \cdot \nabla$,

$$\int_{\mathbb{R}^n} \big| \big[|x|^{-1} x \cdot \nabla f \big](x) \big|^2 \, d^n x \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} |x|^{-2} |f(x)|^2 \, d^n x, \quad f \in C_0^\infty(\mathbb{R}^n \backslash \{0\}), \ n \geq 3.$$

The constant $(n-2)^2/4$ is **optimal**.

Logarithmic Refinements of Hardy's Inequality:

As an example we just consider the **Hardy** case: For $\gamma > 0$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, $|x| < \gamma$, introduce iterated logarithms of the form

$$\begin{aligned}
&(-\ln(|x|/\gamma))_0 = 1, \\
&(-\ln(|x|/\gamma))_1 = (-\ln(|x|/\gamma)), \\
&(-\ln(|x|/\gamma))_{k+1} = \ln((-\ln(|x|/\gamma))_k), \quad k \in \mathbb{N},
\end{aligned}$$

and introduce

$$T_y = \nabla + 2^{-1}|x-y|^{-2}\bigg\{(n-2) + \sum_{j=1}^{m}\prod_{k=1}^{j}[(-\ln(|x-y|/\gamma))_k]^{-1}\bigg\}(x-y),$$
$$0 < |x| < r,\ r < \gamma,\ m \in \mathbb{N},\ n \in \mathbb{N},\ n \geq 2.$$

Logarithmic Refinements of Hardy's Inequ. (contd.):

With T_y^+ the formal adjoint of T_y, one obtains for $f \in C_0^\infty(B_n(y;r)\backslash\{y\})$ (with $B_n(x_0;r_0)$ the open ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $r_0 > 0$)

$$\begin{aligned}(T_y^+ T_y f)(x) &= (-\Delta f)(x) - 4^{-1}|x-y|^{-2}\bigg\{(n-2)^2 \\ &\quad + \sum_{j=1}^{m}\sum_{k=1}^{j}[(-\ln(|x-y|/\gamma))_k]^{-2} f(x)\bigg\} f(x), \quad m \in \mathbb{N},\ n \in \mathbb{N},\ n \geq 2.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{B(y;r)} |(\nabla f)(x)|^2\, d^n x &\geq \frac{1}{4}\int_{B(y;r)} |x-y|^{-2}\bigg\{(n-2)^2 \\ &\quad + \sum_{j=1}^{m}\prod_{k=1}^{j}[(-\ln(|x-y|/\gamma))_k]^{-2}\bigg\}|f(x)|^2\, d^n x, \\ &\quad 0 < r < \gamma,\ f \in C_0^\infty(B(y;r)\backslash\{y\}),\ m \in \mathbb{N}\cup\{0\},\ n \in \mathbb{N},\ n \geq 2.\end{aligned}$$

Logarithmic Refinements of Hardy's Inequ. (contd.):

Explicitly,

$$\int_{B(y;r)} |(\nabla f)(x)|^2 \, d^n x \geq \int_{B(y;r)} \bigg\{ \frac{(n-2)^2}{4|x-y|^2} + \frac{1}{4|x-y|^2[(-\ln(|x-y|/\gamma))]^2} + \frac{1}{4|x-y|^2[(-\ln(|x-y|/\gamma))]^2[\ln(-\ln(|x-y|/\gamma))]^2} + \cdots\cdots \bigg\} |f(x)|^2 \, d^n x,$$

$$0 < r < \gamma, \; f \in C_0^\infty(B(y;r)\backslash\{y\}), \; m \in \mathbb{N} \cup \{0\}, \; n \in \mathbb{N}, \; n \geq 2.$$

Again, this extends to arbitrary open bounded sets $\Omega \subset \mathbb{R}^n$ as long as γ is chosen sufficiently large (e.g., larger than the diameter of Ω). The constants $(n-2)^2/4$ and $1/4$ are **optimal**.

Applications to Lower Semiboundedness and Form Boundedness:

In their simplest form, these inequalities focus on $\mathbb{R}^n \backslash \{0\}$ or $\Omega \backslash \{x_0\}$, $\Omega \subset \mathbb{R}^n$ open and bounded, $x_0 \in \Omega$, etc., and yield sufficient conditions for semiboundedness from below for L^2-realizations of strongly singular differential expressions of the form

$$(-\Delta)^m + V(x), \quad m \in \mathbb{N},\ x \in \mathbb{R}^n \backslash \{0\} \ (\text{or } x \in \Omega \backslash \{x_0\}).$$

However, this represents just the tip of the iceberg and much more is possible: As long as there are countably many singularities, all uniformly separated from each other by some fixed distance $\varepsilon_0 > 0$ (e.g., the singularities could define a lattice), one can localize around each singularity and thus obtain semiboundedness (and self-adjointness) for the entire system with countably many such singularities. This idea of localizing, going back to **J. D. Morgan**, JOT **1**, 109–115 (1979), has recently again been used in

[GMNT16]: F.G., M. Mitrea, I. Nenciu, and G. Teschl, Adv. Math. **301**, 1022–1061 (2016).

Applications to Lower Semiboundedness and Form Boundedness (contd.):

We will aim at $(-\Delta)^2 + W$, where W has countably many strong singularities.

Theorem 9 ([GMNT16], abstracting Morgan, JOT 1, 109–115 (1979))

Suppose that T, W are self-adjoint operators in $\mathcal{H}$ such that $\operatorname{dom}\left(|T|^{1/2}\right) \subseteq \operatorname{dom}\left(|W|^{1/2}\right)$, and let $c, d \in (0,\infty)$, $e \in [0,\infty)$. Moreover, suppose $\Phi_j \in \mathcal{B}(\mathcal{H})$, $j \in J$, $J \in \mathbb{N}$ an index set, leave $\operatorname{dom}\left(|T|^{1/2}\right)$ invariant, i.e., $\Phi_j \operatorname{dom}\left(|T|^{1/2}\right) \subseteq \operatorname{dom}\left(|T|^{1/2}\right)$, $j \in J$, and satisfy conditions (i)–(iii):

(i) $\sum_{j\in J} \Phi_j^* \Phi_j \leq I_{\mathcal{H}}$.

(ii) $\sum_{j\in J} \Phi_j^* |W| \Phi_j \geqslant c^{-1} |W|$ on $\operatorname{dom}\left(|T|^{1/2}\right)$.

(iii) $\sum_{j\in J} \| |T|^{1/2} \Phi_j f \|_{\mathcal{H}}^2 \leqslant d \| |T|^{1/2} f \|_{\mathcal{H}}^2 + e \|f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}\left(|T|^{1/2}\right)$.

Then,

$$\big\| |W|^{1/2} \Phi_j f \big\|_{\mathcal{H}}^2 \leqslant a \big\| |T|^{1/2} \Phi_j f \big\|_{\mathcal{H}}^2 + b \|\Phi_j f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(|T|^{1/2}),\ j \in J,$$

implies

$$\big\| |W|^{1/2} f \big\|_{\mathcal{H}}^2 \leqslant a\, c\, d \big\| |T|^{1/2} f \big\|_{\mathcal{H}}^2 + [a\, c\, e + b\, c] \|f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(|T|^{1/2}).$$

Applications to Lower Semiboundedness and Form Boundedness (contd.):

Thus, the key for applications would be to have c and d arbitrarily close to 1 such that if $a < 1$, also $acd < 1$.
If W is **local** and Φ_j represents the operator of multiplication with **bump functions** ϕ_j, $j \in J \subseteq \mathbb{N}$, such that ϕ_j, $j \in J$ is a family of smooth, real-valued functions defined on $\mathbb{R}^n$ satisfying that for each $x \in \mathbb{R}^n$, there exists an open neighborhood $U_x \subset \mathbb{R}^n$ of x such that there exist only finitely many indices $k \in J$ with $\operatorname{supp}(\phi_k) \cap U_x \neq \emptyset$ and $\phi_k|_{U_x} \neq 0$, as well as

$$\textstyle\sum_{j\in J} \phi_j(x)^2 = 1, \quad x \in \mathbb{R}^n$$

(the sum over $j \in J$ being finite). Then Φ_j and W commute and hence

$$\sum_{j\in J} \Phi_j^* \Phi_j = I_{\mathcal{H}} \text{ and } \sum_{j\in J} \Phi_j^* |W| \Phi_j = |W| \text{ on } \operatorname{dom}\big(|T|^{1/2}\big)$$

yield condition (i) and also (ii) with $c = 1$. (So that takes care of c).

What about d? We'll show next that for all $\varepsilon > 0$, one can choose $d = 1 + \varepsilon$:

Applications to Lower Semiboundedness and Form Boundedness (contd.):

Example. $m = 2$, $T = (-\Delta)^2$, $\operatorname{dom}(T) = H^4(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, $n \geq 5$, and assume that $\operatorname{dom}\big(|T|^{1/2}\big) \subseteq \operatorname{dom}\big(|W|^{1/2}\big)$ (relative form boundedness). Then for arbitrary $\varepsilon > 0$, also condition (iii) holds with $d = 1 + \varepsilon$ as long as

$$\sum_{j\in J} \phi_j(\cdot)^2 = 1, \quad \Big\| \sum_{j\in J} |\nabla \phi_j(\cdot)|^2 \Big\|_{L^\infty(\mathbb{R}^n)} < \infty, \quad \Big\| \sum_{j\in J} |(\Delta \phi_j)(\cdot)|^2 \Big\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

To verify this, one observes that for all $\varepsilon > 0$,

$$\begin{aligned} \sum_{j\in J} \big\| |T|^{1/2}(\phi_j f) \big\|^2_{L^2(\mathbb{R}^n)} &= \sum_{j\in J} \int_{\mathbb{R}^n} |\Delta(\phi_j f)(x)|^2 \, d^n x \\ &\leq (1+\varepsilon) \int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x + C_\varepsilon \|f\|^2_{L^2(\mathbb{R}^n)}, \end{aligned}$$

thus, $d = 1 + \varepsilon$ in condition (iii).

Applications to Lower Semiboundedness and Form Boundedness (contd.):

This follows from the elementary estimate (for some constant $C_\varepsilon \in (0,\infty)$):

$$\begin{aligned}
\sum_{j\in J} \int_{\mathbb{R}^n} |\Delta(\phi_j f)(x)|^2 \, d^n x &\leq \int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \quad \bigg(\longleftrightarrow \sum_{j\in J} \phi_j(\cdot)^2 = 1 \bigg) \\
&\quad + \bigg\| \sum_{j\in J} |(\Delta\phi_j)(\cdot)|^2 \bigg\|_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)} \\
&\quad + 4 \bigg\| \sum_{j\in J} |(\Delta\phi_j)(\cdot)| |(\nabla\phi_j)(\cdot)| \bigg\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |(\nabla f)(x)| |f(x)| \, d^n x \\
&\quad + 2 \bigg\| \sum_{j\in J} |(\Delta\phi_j)(\cdot)| |\phi_j(\cdot)| \bigg\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |(\Delta f)(x)| |f(x)| \, d^n x \\
&\quad + 4 \bigg\| \sum_{j\in J} |\phi_j(\cdot)| |(\nabla\phi_j)(\cdot)| \bigg\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |(\nabla f)(x)| |(\Delta f)(x)| \, d^n x \\
&\leq (1+\varepsilon) \int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x + C_\varepsilon \|f\|^2_{L^2(\mathbb{R}^n)}, \quad f \in H^2(\mathbb{R}^n).
\end{aligned}$$

Applications to Lower Semiboundedness and Form Boundedness (contd.):

Strongly singular potentials W that are covered by Theorem 9 are, e.g., of the following form: Let $J \subseteq \mathbb{N}$ be an index set, and $\{x_j\}_{j\in J} \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 5$, be a set of points such that

$$\inf_{\substack{j,j'\in J\\ j\neq j'}} |x_j - x_{j'}| > 0 \quad \text{(e.g., a **lattice of points** ....).}$$

Let ϕ be a nonnegative smooth function which equals 1 in $B_n(0;1/2)$ and vanishes outside $B_n(0;1)$. Let $\sum_{j\in J} \phi(x - x_j)^2 \geqslant 1/2$, $x \in \mathbb{R}^n$, and set $\phi_j(x) = \phi(x - x_j)\big[\sum_{j'\in J} \phi(x - x_{j'})^2\big]^{-1/2}$, $x \in \mathbb{R}^n$, $j \in J$, such that $\sum_{j\in J} \phi_j(x)^2 = 1$, $x \in \mathbb{R}^n$. In addition, let $\gamma_j \in \mathbb{R}$, $j \in J$, $\gamma, \delta \in (0,\infty)$ with

$$|\gamma_j| \leq \gamma < \big[n(n-4)/4\big]^2, \quad j \in J \quad \text{(the **Rellich** constant ....),}$$

and consider

$$\boxed{W_0(x) = \sum_{j\in J} \gamma_j\, |x - x_j|^{-4}\, e^{-\delta|x - x_j|},} \quad x \in \mathbb{R}^n \backslash \{x_j\}_{j\in J}.$$

Then by **Rellich's inequality** in $\mathbb{R}^n$, $n \geq 5$, W_0 is **form bounded** with respect to $T = (-\Delta)^2$ with form bound **strictly less than one**.

Some Extensions:

Current joint work with **Michael Pang** focuses on **"radial extensions"**: Recalling the **improved/refined Hardy inequality** involving the **radial derivative**, $\big(|x|^{-1}x \cdot \nabla f\big)(x) := \partial f/\partial r(x)$, if $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, $n \geq 3$, then

$$\int_{\mathbb{R}^n} \big|(\nabla f)(x)\big|^2 \, d^n x \geq \underbrace{\int_{\mathbb{R}^n} \left|\frac{\partial f}{\partial r}(x)\right|^2 d^n x}_{\textbf{improvement}} \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} |x|^{-2} |f(x)|^2 \, d^n x.$$

Thus, we conjectured, and then proved, that also the **Rellich** inequality (in fact, the entire sequence of **higher-order Hardy–Rellich** inequalities) extends in this radial context:

Some Extensions:

E.g., if $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, $n \in \mathbb{N}$, $n \geq 5$, then,

$$\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 \, d^n x \geq \underbrace{\int_{\mathbb{R}^n} \left| \frac{\partial^2 f}{\partial r^2}(x) + \frac{n-1}{|x|} \frac{\partial f}{\partial r}(x) \right|^2 d^n x}_{\textbf{improvement}}$$

$$\geq \left[\frac{n(n-4)}{4} \right]^2 \int_{\mathbb{R}^n} |x|^{-4} |f(x)|^2 \, d^n x.$$

Indeed, **Machihara, Ozawa, Wadade**, Math. Z. **286**, 1367–1373 (2017), just published this. Still, we have a different proof and further extensions ......