Comprehensive Exam – Analysis (June 2009)
Answer all questions

1. Prove the following statements:
(a) Every convergent sequence of real numbers is bounded.

(b) Suppose $f(x)$ is differentiable and $f'(x)$ is continuous on $[1, \infty)$. If both $\int_1^\infty f(x)\,dx$ and $\int_1^\infty f'(x)\,dx$ converge, then show that $\lim_{x\to\infty} f(x) = 0$.

2. (a) State and prove the Weierstrass M-test for uniform convergence of a series $\sum_{n=0}^{\infty} a_n(x)$ on an interval $I \subseteq \mathbf{R}$.

(b) Use the Weierstrass M-test to show that the series

$$\sum_{n=0}^{\infty} \frac{\sin(nx)}{1+n^2}$$

represents a continuous function on $\mathbf{R}$.

3. Give examples of each of the following.
(a) A function sequence $f_n(x) : [0,1] \to \mathbf{R}$ such that the sequence converges to 0 pointwise, but $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq 0$.

(b) A sequence of differentiable functions $f_n(x) : [0,1] \to \mathbf{R}$, which converges uniformly, but the sequence of derivatives $f_n'(x)$ does *not* converge pointwise on all of $[0,1]$.

4. (a) State the Stone-Weierstrass theorem for the set $C(X)$ of continuous functions on a compact metric space X.

(b) Use the Stone-Weierstrass theorem to show that the set of functions of the form

$$h(x) = \sum_{j=1}^{n} a_j e^{b_j x}\,, \quad a_j,\, b_j \in \mathbf{R}\,,\ j = 1, \ldots, n\,, \quad n \in \mathbf{N}\,,$$

is dense in $C([a,b])$.