PhD Comprehensive Exam – Ring Theory (May 2023)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

In any question for which an example of a ring or module having certain properties is requested, make sure to justify why the example you've given really does have those properties. (In other words, don't just write down the example without appropriate explanation.)

Throughout, R, S, and T denote associative rings with identity. Homomorphisms of left modules will be written on the right: so we write $(m)f$, and fg means 'first f, then g'.

1. Modules. Let M_i $(i \in I)$ be nonzero left R-modules.

(a) Show that if each M_i is injective, then so is $\prod_{i\in I} M_i$.

(b) Show that if $\prod_{i\in I} M_i$ is noetherian, then each M_i is noetherian and I is finite.

2. Matrix Rings.

(a) Show that for any positive integer n, a matrix belongs to the center of $\mathbb{M}_n(R)$ if and only if it is of the form rI_n, where r is in the center $Z(R)$ of R, and I_n is the identity matrix.

(b) Suppose that K_1, K_2 are fields, and n_1, n_2 are positive integers. Show that if $\mathbb{M}_{n_1}(K_1) \cong \mathbb{M}_{n_2}(K_2)$, then $K_1 \cong K_2$ and $n_1 = n_2$.

(c) Give an example of rings R_1, R_2 and positive integers n_1, n_2 such that $R_1 \not\cong R_2$ but $\mathbb{M}_{n_1}(R_1) \cong \mathbb{M}_{n_2}(R_2)$.

3. Jacobson Radical.

(a) Show that for any rings R_i $(i \in I)$, $\mathrm{rad}(\prod_{i\in I} R_i) = \prod_{i\in I} \mathrm{rad}(R_i)$.

(b) Explicitly describe all left artinian rings R with $\mathrm{rad}(R) = 0$.

(c) The *socle* $\mathrm{soc}(M)$ of a left R-module M is defined to be sum of all simple submodules of M. Show that

$$\mathrm{soc}(M) \subseteq \{m \in M \mid \mathrm{rad}(R) \cdot m = 0\},$$

with equality if $R/\mathrm{rad}(R)$ is a left artinian ring.

4. Functors and natural transformations.

(a) Let $e = e^2$ be an idempotent in R. Let M be any left R-module. For each $exe \in eRe$ and $\varphi \in \mathrm{Hom}_R(Re, M)$, define the element $exe * \varphi$ of $\mathrm{Hom}_R(Re, M)$ in such a way that $*$ makes $\mathrm{Hom}_R(Re, M)$ into a left eRe-module. Show all the details.

(b) The assignment $F : R - \mathrm{Mod} \to eRe - \mathrm{Mod}$ given by setting $F(M) = \mathrm{Hom}_R(Re, M)$ for each left R-module M, and setting $F(\psi) = \psi_*$ for each $\psi \in \mathrm{Hom}_R(M, N)$, gives an additive covariant functor from $R - \mathrm{Mod}$ to $eRe - \mathrm{Mod}$.

The assignment $G : R{-}\mathrm{Mod} \to eRe{-}\mathrm{Mod}$ given by setting $G(M) = eR \otimes_R M$, and $G(\tau) = \iota_{eR} \otimes \tau$ for each $\tau \in \mathrm{Hom}_R(M, N)$, gives an additive covariant functor from $R - \mathrm{Mod}$ to $eRe - \mathrm{Mod}$. (Here ι_{Re} is the identity homomorphism from Re to itself.)

Prove that F and G are naturally isomorphic functors. Show all the details.

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5. Leavitt path algebras and related ideas.

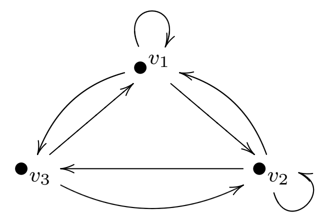
(a) Let $\delta_{i,j}$ denote the Kronecker delta function as usual: $\delta_{i,j} = 1$ if $i = j$; $\delta_{i,j} = 0$ if $i \neq j$. Prove that the following two statements are equivalent for any ring R.

1. There exist $x_1, x_2, x_3, y_1, y_2, y_3 \in R$ for which $y_i x_j = \delta_{i,j} 1_R$ for all $1 \leq i, j \leq 3$, and $x_1 y_1 + x_2 y_2 + x_3 y_3 = 1_R$.

2. $R \cong R^3$ as left R-modules.

For the remainder of this question, let E denote the graph

$E :=$



(b) Compute 'directly' the monoid M_E. (First find a set of representatives of the equivalence classes in M_E, then prove that these equivalence classes are distinct.

(c) Using the theorem of Ara / Moreno / Pardo, we know that $M_E \cong \mathcal{V}(L_K(E))$. Under this identification, which element of M_E corresponds to $[1_{L_K(E)}]$?

(d) Find the matrix $I - A_E^t$, and compute its Smith normal form. Show that your answer to part (d) is consistent with your answer to part (b).

(e) True or False: $L_K(E) \cong L_K(1, 4)$. Fully justify.

6. Equivalence of module categories. (All functors in this question are assumed to be additive and covariant.)

(a) If F is a category equivalence from $R-\mathrm{Mod}$ to $S-\mathrm{Mod}$ then F can be described "concretely" as a specific type of functor, involving a specific bimodule. Give this concrete description of F, and justify how this is achieved.

(b) The left R-module R has many module-theoretic properties. Which three specific module-theoretic properties of R play the central role in any category equivalence between $R - \mathrm{Mod}$ and $S - \mathrm{Mod}$ for some ring S?

(c) If $e = e^2 \in R$ is *any* idempotent, then the left ideal Re necessarily has two of the three properties you listed in 6(b). Give a necessary and sufficient ring-theoretic condition on e so that Re has the third of the three module-theoretic properties you gave in the previous question.

(d) Referring to 6(c), denote the ring-theoretic property by $\mathcal{T}$, and denote the module-theoretic property by $\mathcal{P}$. Prove that for any idempotent $e \in R$, e has property $\mathcal{T}$ if and only if Re has property $\mathcal{P}$.

(e) Give a necessary and sufficient ring-theoretic condition relating two rings R and S, which is equivalent to the statement: the categories $R - \mathrm{Mod}$ and $S - \mathrm{Mod}$ are equivalent. (You need not prove this. To rephrase the question: give a ring-theoretic condition equivalent to the statement "R and S are Morita equivalent.")