PhD Preliminary Exam – Linear Algebra (January 2024)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Let $\mathcal{C}$ be a family of sets. Define what it means for $\mathcal{C}$ to be a **chain**.

(b) Let $\mathcal{C}$ be a chain. Prove by induction that for every positive integer n: if A is a set with n elements such that $A \subseteq \bigcup \mathcal{C}$, then there is some $C \in \mathcal{C}$ such that $A \subseteq C$.

(c) State Zorn's Lemma.

(d) Use Zorn's Lemma to prove that every vector space V over a field F has a basis.

2. Let V and W be vector spaces over a field F. Suppose that $T\colon V \to W$ is linear.

(a) Define what it means for T to be **invertible**.

(b) Prove that T is invertible if and only if T is one-to-one and onto.

(c) Suppose that $g\colon W \to V$ is a function such that $T \circ g = I_W$ and $g \circ T = I_V$. Prove that g is linear.

3. Let V and W be finite-dimensional vector spaces over a field F and $T\colon V \to W$ be linear.

(a) If β and γ be ordered bases for V and W respectively, define the matrix representation $[T]_\beta^\gamma$ of T relative to the bases β and γ.

(b) Suppose that $\dim(V) = \dim(W)$. Prove that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

4. Let $V = P_2(\mathbb{R})$ and $T\colon V \to V$ be defined by $T(ax^2 + bx + c) = cx^2 + bx + a$.

(a) Let $\beta = (1, x, x^2)$ be an ordered basis. Compute $[T]_\beta$

(b) Determine whether T is diagonalizable or not.

(c) If T is diagonalizable, find a basis γ for V such that $[T]_\gamma$ is a diagonal matrix. Compute $[T]_\gamma$.

5. Let V be a finite dimensional, inner product space and let T be a self-adjoint operator on V.

(a) Prove that every eigenvalue of T is real.

(b) Prove that there exists a self-adjoint operator C on V such that $T = C \circ C \circ C$.

(c) Prove that there exists a self-adjoint operator S on V such that $T = S \circ S$ if and only if $\langle T(x), x\rangle \geq 0$ for all $x \in V$.

6. Let V be an inner product space over a field F.

(a) Let W be a finite-dimensional subspace of V. If $x \notin W$, then there exists some $y \in V$ such that $y \in W^{\perp}$ but $\langle x, y \rangle \neq 0$. Note: you may assume that $V = W + W^{\perp}$ without proof.

(b) Let T be a linear operator on V.
(i) Prove that $R(T^*)^{\perp} = N(T)$.
(ii) If V is finite dimensional, then show that $R(T^*) = N(T)^{\perp}$.