PhD Preliminary Exam – Linear Algebra (January 2023)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Consider a vector space V over a field F. Suppose we remove the axiom that $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ for every $\mathbf{v},\mathbf{w}\in V$. Prove this axiom from the remaining axioms for a vector space (Hint: expand $(1+1)(\mathbf{v}+\mathbf{w})$ in two different ways).

(b) Circle all of the following which are true. No justification is needed.

1. For any field F, there exists some vector space V over F.
2. The empty set is a basis for some vector space.
3. Every linearly independent set in a vector space is a subset of some linearly dependent set.
4. Every vector space has a basis.
5. For any field F, there exists some infinite-dimensional vector space V over F.
6. Any two bases of a finite-dimensional vector space have the same cardinality (size).

(c) Let V and W be finite-dimensional vector spaces over a field F. Suppose there exist injective (one-to-one) linear maps $T_1\colon V\to W$ and $T_2\colon W\to V$. Prove that $\dim(V)=\dim(W)$.

2. If S_1 and S_2 are nonemply subsets of a vector space V, then the sum S_1+S_2 is defined by the set $\{\mathbf{x}+\mathbf{y}:\mathbf{x}\in S_1,\mathbf{y}\in S_2\}$. Suppose W_1 and W_2 are subspaces of a vector space V over a field F.
(a) Prove that W_1+W_2 is a subspace of V containing both W_1 and W_2.

(b) If W_1 and W_2 are finite dimensional then prove that W_1+W_2 is also finite-dimensional, and $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)-\dim(W_1\cap W_2)$.

3. Let F be a field and let A,B,C be any $n\times n$ matrices with entries in F.
(a) Prove that the map $L_A\colon F^n\to F^n$ defined by $L_A(\mathbf{v})=A\mathbf{v}$, $\mathbf{v}\in F^n$ is a linear map.

(b) Determine $[L_A]_\beta$ where $\beta=\{\mathbf{e_1},\dots,\mathbf{e_n}\}$ is the standard, ordered basis for F^n.

(c) If $L_A=L_B$, then prove that $A=B$.

(d) Prove that $L_A\circ L_B=L_{AB}$ where $\circ$ denotes the composition of the linear maps. Do NOT use associativity: $A(B\mathbf{v})=AB(\mathbf{v})$. (Hint: Show that the maps agree on a basis of F^n).

(e) Assume that the composition of linear maps is associative and use the above results to prove that $(AB)C=A(BC)$. (Hint: Use parts (c) and (d)).

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4. Let A be an $n \times n$ matrix satisfing $A^2 = I$, where I is the identity matrix.

(a) Find all the eigenvalues of A.

(b) Prove that $\text{rank}(A + I) + \text{rank}(A - I) = n$.

(c) Prove that A is diagonalizable.

5. Let $\mathbf{x}$ be a unit vector in $\mathbb{C}^n$ and define the $n \times n$ matrix $H = I_n - 2\mathbf{x}\mathbf{x}^*$ where $\mathbf{x}^*$ denotes the transpose conjugate of $\mathbf{x}$.

(a) Show that $H\mathbf{x} = -\mathbf{x}$.

(b) If $\mathbf{y}$ is orthogonal to $\mathbf{x}$ then show that $H\mathbf{y} = \mathbf{y}$.

(c) Show that the matrix H is both Hermitian and unitary.

(d) Suppose $L_P : \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $L_P(\mathbf{v}) = P\mathbf{v}$, $\mathbf{v} \in \mathbb{C}^n$ where $P = I - \mathbf{x}\mathbf{x}^*$. Show that L_P is the orthogonal projection of $\mathbb{C}^n$ onto the subspace $(span(\mathbf{x}))^\perp$.

6. Define a linear transformation on $\mathbb{C}^4$ by $T(w_1, w_2, w_3, w_4) = (0, w_2 + w_4, w_3, w_4)$.

(a) Determine the characteristic polynomial of T.

(b) Determine the minimal polynomial of T.

(c) Determine the Jordan canonical form of T.