PhD Comprehensive Exam – Complex Analysis (August 2026)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

1. Recall that a function $f(z) = u(x, y) + iv(x, y)$ defined on an open set $S \in \mathbb{C}$ is *differentiable* at a point $z_0 \in S$ if the first partial derivatives are continuous and satisfy the Cauchy-Riemann equations at z_0. f is *analytic* at z_0 if it is differentiable in some open set containing z_0.

(a) Show that $f(z) = \cos(\overline{z})$ is differentiable at infinitely many points in $\mathbb{C}$, yet it is *nowhere* analytic.

(b) Consider the function $f(z) = \sqrt{z^2 + 1} = \sqrt{(z - i)(z + i)}$. Define

$$z - i = r_1 e^{i\theta_1}\,, \quad z + i = r_2 e^{i\theta_2}\,,$$

such that $f(z) = (r_1 r_2)^{1/2} e^{i\Theta/2}$, where $\Theta = \theta_1 + \theta_2$.

(i) Show that $z = \pm i$ are branch points of $f(z)$.

(ii) Restrict $-\frac{\pi}{2} \leq \theta_1, \theta_2 < \frac{3\pi}{2}$. Next prove that $[-i, i]$ is a branch cut by showing that along the imaginary axis, $f(z)$ is continuous at each point $z = iy$, $|y| > 1$ while $f(z)$ has a jump discontinuity at each point $z = iy$, $|y| < 1$. (Hint: Calculate $\lim_{z \to iy} \Theta(z)$ in each case).

2. Suppose a function $f(z)$ is analytic for all z such that $|z| > R > 0$ except possibly at $z = \infty$. Then its behavior near $z = \infty$ is determined by the behavior of the function $F(t) = f(1/t)$ on the annulus $0 < |t| < 1/R$.

(a) Show that an entire function has a removable singularity at $z = \infty$ if and only if it is a constant.

(b) Suppose an entire function satisfies $|f(z)| \leq A|z|^n$, $A > 0$ for all z such that $|z| > R > 0$. Prove that f is a polynomial of degree $\leq n$.

(c) Use part (b) to prove that an entire function has a pole of order n at $z = \infty$ if and only if it is a polynomial of degree n.

3. (a) Use contour integration to show that

$$\int_0^{2\pi} \frac{1}{1 + \sin^2 x}\, dx = \pi\sqrt{2}\,.$$

Useful identities: $\sqrt{3 \pm 2\sqrt{2}} = \sqrt{2} \pm 1$.

(b) Use contour integration (justifying all steps) to show that

$$\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\, dx = \frac{\pi}{2}(b - a)\,, \quad a, b > 0\,.$$

(Hint: (i) Consider $f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}$ (ii) Use $\sin x \geq \frac{2x}{\pi}$, $0 \leq x \leq \frac{\pi}{2}$ (Jordan's Lemma)).

4. This problem relates to the fact that *zeros of an analytic function are isolated.*

(a) If $f(z)$ is analytic at z_0 and $f(z_0) = 0$ then prove that there exists $r > 0$ such that either $f(z) = 0$ OR $f(z) \neq 0$ in the deleted neighborhood: $0 < |z - z_0| < r$.

(b) Suppose $f(z)$ is analytic in a region S, and $\{z_n\}$ is an infinite sequence of points in S with $z_n \to z_0 \in S$. If $f(z_n) = 0$ for all n then prove that $f(z)$ is *identically* zero on S.

(c) Suppose $C(z), D(z)$ are entire functions which satisfy $C^2(z) + S^2(z) = 1 + i$ on the unit circle $|z| = 1$, show that $C^2(z) + S^2(z) = 1 + i$ for *all* $z \in \mathbb{C}$.

5. In this problem the following version of Schwarz Lemma is useful: *If $f(z)$ is analytic on the open unit disk D with $|f(z)| \leq 1,\ z \in D$ and $f(0) = 0$, then $|f(z)| \leq |z|$ on D. Moreover, if $|f(z)| = |z|$ for some $z \neq 0$ in D then $f(z) = cz$ on D with $|c| = 1$.* No need to prove it.

(a) Let $f : D \to D$ be a conformal map such that $f(0) = 0$. Prove that f is a rotation, that is, $f(z) = cz,\ |c| = 1$. Do NOT use part (c) to prove this.

(b) Let $\psi_\alpha : D \to D$ be defined by

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}\,, \quad \alpha \in D\,.$$

Show that ψ_α is a conformal map (automorphism) of the unit disk.

(c) Let f be *any* automorphism of the unit disk D. Prove that there exists $\theta \in \mathbb{R}$ and $\alpha \in D$ such that

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z}\,.$$

6. (a) Let $f = u + iv$ be analytic on the closed unit disk $\overline{D}$. Show that for any $z \in D$ where D is the open unit disk

$$2\pi i f(z) = \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_C \frac{f(\zeta)}{\zeta - 1/\overline{z}}\, d\zeta\,,$$

where C is the unit circle bounding D.

Deduce the Poisson integral formula for the harmonic function $u = \mathrm{Re}(f)$

$$u(r, \theta) = \frac{1}{2\pi i} \int_C \frac{1 - |z|^2}{|\zeta - z|^2}\, u(\zeta)\, \frac{d\zeta}{\zeta} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}\, u(\phi)\, d\phi\,,$$

where $z = re^{i\theta}, \zeta = e^{i\phi}$.

(b) Compute $u(r, \theta)$ in part (a) if $u(\phi) = \sin\phi$. (Hint: Use the variables z, ζ and a suitable contour integral over the unit circle $C : \zeta\overline{\zeta} = 1$.)