PhD Comprehensive Exam – Complex Analysis (August 2024)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

1. Let $U \subseteq \mathbb{C}$ be a nonempty open set and let $f : U \to \mathbb{C}$ be an analytic function with its derivative defined for each $z \in U$ as,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

(a) Denote $z = x + iy$, $x, y \in \mathbb{R}$, $z \in U$ and $f(z) = u(x, y) + iv(x, y)$. Derive the Cauchy-Riemann equations for $u(x, y)$ and $v(x, y)$.

(b) For all $z \in U$ such that $f'(z) \neq 0$, show that

$$(i)\; u_x^2 + u_y^2 = v_x^2 + v_y^2 = |f'(z)|^2 \qquad\qquad (ii)\; u_x v_x + u_y v_y = 0\,,$$

where the subscripts denote partial derivatives.

(c) Suppose that a function $f(z) = u(x, y) + iv(x, y)$ and its conjugate $\overline{f(z)} = u(x, y) - iv(x, y)$ are both analytic in a connected domain U. Show that $f(z)$ must be constant throughout U.

2. If a function $f(z)$ is holomorphic for $|z| > R$ for large $R > 0$ then its behavior near $z = \infty$ is determined by the behavior of the function $F(t) = f(1/t)$ near $t = 0$. Thus $f(z)$ is holomorphic or meromorphic at $z = \infty$ if $F(t)$ has the corresponding behavior $t = 0$.
(a) Use a Laurent series expansion around $t = 0$ to determine if each of the following functions are holomorphic or meromorphic at $z = \infty$

$$(i)\; f(z) = \frac{z^2 + z + 1}{z^2 - 1} \qquad\qquad (ii)\; f(z) = z\cos(1/z)\,.$$

(b) Prove that a meromorphic function in the extended complex plane $\mathbb{C} \cup \{\infty\}$ has
(i) finitely many poles in $\mathbb{C}$, and
(ii) is a rational function. (Hint: Use Liouville's Theorem).

3. (a) Use contour integration to show that for $0 \leq a^2 < 1$

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1 - a^2}}\,.$$

(b) Use contour inegration justifying all steps to show that

$$\int_0^{\infty} \frac{x^{k-1}}{x + a} dx = \frac{\pi a^{k-1}}{\sin k\pi} \quad 0 < k < 1,\; a > 0\,.$$

(Hint: Take a branch cut along the positive real axis and use a keyhole (around $z = 0$) contour).

over

4. Rouche's theorem states: *Suppose f, g are holomorphic functions on an open set containing a simple, piecewise smooth, closed contour γ and its interior. If $|f(z)| > |g(z)|$ for all $z \in \gamma$ then f and $f+g$ have the same number of zeros (including multiplicities) inside the contour γ.*

(a) Prove Rouche's theorem using the Argument principle.

(b) Use Rouche's theorem to show that $az^n = e^z$, $n \in \mathbb{N}$ has n roots inside the unit circle $|z| = 1$, if $a > e$.

(c) Suppose f is a non-constant, holomomorphic function on an open set containing the closed unit disk $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. If $|f(z)| = 1$ on $|z| = 1$, then show that
(i) $f(z) = 0$ has a solution for $|z| < 1$ (Hint: Use the Maximum modulus principle).
(ii) The image of f contains $\overline{D}$ (Hint: Use Rouche's theorem).

5. Schwarz Lemma states: *If $f(z)$ is analytic on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ with $|f(z)| \leq 1$ and $f(0) = 0$, then $|f(z)| \leq |z|$ on D and $f'(0) \leq 1$. Moreover, if $|f(z)| = |z|$ for some $z \in D$, $z \neq 0$, or if $f'(0) = 1$, then $f(z) = cz$ on D where c is a constant with $|c| = 1$.*

(a) Prove Schwarz Lemma.

(b) Let $D_r = \{z \in \mathbb{C} : |z| < r\}$ and $z_0 \in D_r$. Prove that $\zeta = T(z) = \dfrac{r(z - z_0)}{r^2 - \overline{z}_0 z}$ is a conformal map of D_r onto the unit disk D with $T(z_0) = 0$.

(c) Suppose $f : D_r \to \mathbb{C}$ be holomorphic with $|f(z)| \leq M$, $M > 0$. Let $w_0 = f(z_0)$, $z_0 \in D_r$. Use part (b) and Schwarz Lemma to show that

$$(i)\ \left|\frac{M(f(z) - w_0)}{M^2 - \overline{w}_0 f(z)}\right| \leq \left|\frac{r(z - z_0)}{r^2 - \overline{z}_0 z}\right| \qquad (ii)\ \frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \quad \text{if} \quad M = r = 1\,.$$

6. (a) State precisely the Riemann mapping theorem including the conditions that make the mapping unique. Then prove the uniqueness of the mapping (not the full theorem).

(b) Prove that the following infinite product converges if $|z| < 1$

$$P(z) = (1+z)(1+z^2)(1+z^4) + \cdots = \prod_{k=0}^{\infty}(1 + z^{2^k})\,, \quad \text{and show that} \quad P(z) = \frac{1}{1-z}\,.$$

(c) Given the infinite froduct formula: $\dfrac{\sin \pi z}{\pi} = z \displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$

(i) Take the second derivative of the log of the expression above to derive the formula

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}\,.$$

(ii) Equate the *constant* terms of Laurent series of both sides of the above expression to derive

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$