Comprehensive Exam - Analysis (June 2026)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) Let $f(x) = \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(0) = \frac{1}{2}$, yet f is not increasing in any neighborhood of $x = 0$.

(b) In contrast to part (a), show that $g(x) = 2x + x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = 0$, is increasing in some neighborhood of $x = 0$.

2. (a) Prove the mean value theorem for integrals: Suppose the function f is continuous on $[a, b]$. Then there is some $c \in (a, b)$ that satisfies $\int_a^b f(x)dx = (b - a)f(c)$.

(b) Prove a converse to the mean value theorem for integrals: Let f be a strictly increasing, continuous function on $[a, b]$ and suppose $c \in (a, b)$. Then there are α, β with $a \leq \alpha < c < \beta \leq b$ that satisfy

$$\int_\alpha^\beta f(x)dx = (\beta - \alpha)f(c)$$

A suggestion: You may suppose $f(c) = 0$ for otherwise you could consider $f(x) - f(c)$ instead of $f(x)$.

(c) Consider the function $f(x) = x$ on the interval $[0, 1]$. Show that for $c = 1/2$ there are an infinite number of subintervals (α, β) that satisfy the converse in part (b).

3. Define the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = nxe^{-nx}$ and the sequence $g_n : [0, 1] \to \mathbb{R}$ by $g_n(x) = n^2xe^{-nx}$. Answer the following questions for both f_n and g_n:

(a) The sequence $\{f_n\}$ converges pointwise to the function F. What is F?

(b) The sequence $\{g_n\}$ converges pointwise to the function G. What is G?

(c) Is the convergence uniform in parts (a) and (b)? Show why or why not for each.

(d) Evaluate both $\lim_{n\to\infty} \int_0^1 f_n dx$ and $\int_0^1 \lim_{n\to\infty} f_n \, dx$. Are they equal?

(e) Evaluate both $\lim_{n\to\infty} \int_0^1 g_n dx$ and $\int_0^1 \lim_{n\to\infty} g_n \, dx$. Are they equal?

(f) Give a reason why there is a difference in your answers to parts (d) and (e).

4. (a) Let (X, d) be a metric space. An infinite subset S of X has a *limit point* $p \in X$ if for every $\epsilon > 0$ there is a point $q \in S$ such that $q \neq p$ and $d(p, q) < \epsilon$. Prove that X is sequentially compact if and only if every infinite subset S of X has a limit point in X.

(b) Let $(\mathbb{N}, d)$ be the metric space with the metric $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$. Determine if $(\mathbb{N}, d)$ is sequentially compact. Prove your assertions.

5. Let $X = \mathbb{Q}$ be the metric subspace of $\mathbb{R}$ consisting of all rational numbers.

(a) Determine all real t such that the set $U_t = \{q \in X \mid q < t\}$ is *both* an open and closed in X. Prove your assertion.

(b) Suppose that A is a subspace of X consisting of at least two points. Prove that A is disconnected.

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a non–constant continuous function. Prove that the image $f(\mathbb{Q}) = \{f(q) \mid q \in \mathbb{Q}\}$ *cannot* be a closed set of $\mathbb{R}$. (*Hint*: Show that if $f(\mathbb{Q})$ is closed, then $f(\mathbb{Q}) = f(\mathbb{R})$).

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Suppose that for every point (a, b) in $\mathbb{R}^2$ with $\|(a, b)\| = 1$ we have $f(a, b) = f(0, 0)$.

(a) Prove that for every point (a, b) in $\mathbb{R}^2$ with $\|(a, b)\| = 1$ we have

$$\int_0^1 \left(a\frac{\partial f}{\partial x}(ta, tb) + b\frac{\partial f}{\partial y}(ta, tb) \right) dt = 0 \, .$$

(b) Prove that there are infinitely many points (x, y) in $\mathbb{R}^2$ with $0 < \|(x, y)\| < 1$ such that $\langle \nabla f(x, y), (x, y) \rangle = 0$.

(c) Consider the example function $f(x, y) = (x^2 + y^2 - 1)(x^4 + y^4)$ for all (x, y) in $\mathbb{R}^2$. Find explicitly the set of points (x, y) for which the property of part (b) holds for this example.