Comprehensive Exam – Analysis (June 2025)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Consider the sequence $\{x_n\}$ defined by $x_{n+1} = 1 + 1/x_n$ for all $n \in \mathbb{N}$ with $x_1 = 1$.

(a) Show that $0 < x_{2n+2} \leq x_{2n}$ and $x_{2n+1} \geq x_{2n-1} > 0$ for all $n \in \mathbb{N}$.

(b) Prove that the subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ both converge to the *same* limit $x \in \mathbb{R}$. Determine the value of x.

(c) Prove that the sequence $\{x_n\}$ converges to the limit x found in part (b).

2. (a) Suppose $f(x) = \frac{x}{3} + x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(0) = \frac{1}{3}$. Prove that there is however *no* neighborhood of $x = 0$ where f is increasing.

(b) If $f : [a, b] \to \mathbb{R}$ is integrable then show that the function $F : [a, b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f$ is continuous for all x in $[a, b]$.

3. Define a sequence of functions $\{f_n : [0, 1] \to \mathbb{R}\}$ by $f_n(x) = n^2 x^n (1 - x)$.

(a) Show that $\{f_n\}$ converges pointwise to a function f on $[0, 1]$. Determine f.

(b) Show that $\{f_n\}$ does not converge uniformly on $[0, 1]$.

(c) Evaluate $\lim\limits_{n\to\infty} \int_0^1 f_n$ and $\int_0^1 \lim\limits_{n\to\infty} f_n$. Determine if they are equal.

4. Let (X, d) be a metric space and $A \subseteq X$.

(a) Give a definition of the closure $\bar{A}$ of A. Prove that $\bar{A}$ is the *smallest* closed subset of X containing A.

(b) If (X, d) is a complete metric space then prove that $(\bar{A}, d)$ is complete.

(c) If (X, d) is a compact metric space then prove that $(\bar{A}, d)$ is compact.

over

5. Let (X, d) be a metric space and let $\{x_n\}$ be a convergent sequence in X with $\lim x_n = x$. Prove that the union $\{x\} \cup \{x_n\}$ is a compact subset of X.

(b) Suppose $(X, d), (Y, d')$ are metric spaces and $f : X \to Y$ is a function such that $f|_A$ is continuous on all compact subsets $A \subseteq X$. Prove that f is continuous on X.

6. (a) Let $D \subseteq \mathbb{R}^n$ be an open set. State the Mean Value Theorem for a continuously differentiable function $f : D \to \mathbb{R}$.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \frac{x^2 y}{x^2 + y^2}, \quad (x, y) \neq (0, 0), \quad \text{and} \quad f(0, 0) = 0\,.$$

(i) Show that the first partial derivatives of f exist but are not continuous at $(0, 0)$.

(ii) Show that f still satisfies the Mean Value Theorem in entire $\mathbb{R}^2$ i.e., including $(0, 0)$.