Comprehensive Exam – Analysis (June 2024)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) A sequence is defined recursively by $x_{n+1} = x_n + \frac{1}{x_n}$, for all $n \geq 0$, with $x_0 = 1$. Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

(b) Prove that $\lim\limits_{n\to\infty} \dfrac{1}{(n!)^{1/n}} = 0$. *Hint*: Show that $(n!)^2 = \prod\limits_{k=1}^{n} k(n-k+1) \geq n^n$.

(c) Suppose $a_k \geq 0$ for large k, and $\sum\limits_{k=1}^{\infty} \dfrac{a_k}{k}$ converges. Prove that $\lim\limits_{j\to\infty} \sum\limits_{k=1}^{\infty} \dfrac{a_k}{j+k} = 0$.
Hint: Use Cauchy convergence criterion for series.

2. (a) Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in I prove that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

(b) Suppose S is a *dense* subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ is a uniformly continuous function.
(i) Use part (a) to prove that if $y_n \in S$, $y_n \neq x$, and $\lim\limits_{n\to\infty} y_n = x$, then $L_x = \lim\limits_{n\to\infty} f(y_n)$ exists for *all* $x \in \mathbb{R}$, and is independent of the choice of the sequence $\{y_n\}$.
(ii) Prove that $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = L_x$ is uniformly continuous.

3. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $\varphi''(x) \geq 0$ for all $x \in \mathbb{R}$.

(a) Show that for any $x_0 \in \mathbb{R}$ we have $\varphi(x) \geq \varphi(x_0) + \varphi'(x_0)(x - x_0)$ for all $x \in \mathbb{R}$.

(b) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Show that $\varphi\left(\int_0^1 f(t)\,dt\right) \leq \int_0^1 \varphi(f(t))\,dt$.
Hint: Use part (a) with $x_0 = \int_0^1 f(t)\,dt$.

(c) Apply part (b) to show that $\int_0^1 \sqrt{4t^3+1}\,dt \leq \sqrt{2}$.

4. Suppose $\{f_n : [a,b] \to \mathbb{R}\}$ is a uniformly convergent sequence of continuous functions.
(a) Show that there is a constant $M \geq 0$ such that $|f_n(x)| \leq M$ for all $x \in [a,b]$ and $n \geq 1$.

(b) Suppose g is continuous on $[-M, M]$ with M given in part (a). Define $h_n : [a,b] \to \mathbb{R}$ by $h_n(x) = g(f_n(x))$, $n \geq 1$. Prove that the sequence $\{h_n\}$ converges uniformly on $[a,b]$.

5. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Suppose there is a sequence $\{c_k\}$ of nonnegative constants such that for any $p, q \in X$ we have $d(T^k(p), T^k(q)) \leq c_k d(p, q)$ where $T^k := \underbrace{T \circ \cdots \circ T}_{k \text{ times}}$, and suppose the series $\sum_{k=1}^{\infty} c_k$ converges.

(a) Prove that for any $p \in X$ the sequence $\{p_k := T^k(p)\}$ is a Cauchy sequence in X.

(b) Suppose in addition that X is a complete metric space. Prove that there exists a *unique* point $p_* \in X$ such that $T(p_*) = p_*$.

6. (a) Suppose A is a nonempty subset of $\mathbb{R}^n$. Prove that A is sequentially compact if and only if *every* continuous function $f : A \to \mathbb{R}$ is bounded.

(b) Let (X, d) be a countably infinite discrete metric space, meaning $X = \{p_1, p_2, \ldots\}$ and $d(p_j, p_k) = 1$ for all $j \neq k$. Define $f : X \to \mathbb{R}$ by

$$f(p) = \sum_{k=1}^{\infty} \frac{d(p, p_k)}{2^k}, \quad p \in X\,.$$

Prove that f is a bounded continuous function that attains a minimum but does not attain a maximum value.