Comprehensive Exam – Analysis (Jan 2024)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let $\{x_n\}$, $n \geq 0$ be a sequence defined recursively by $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$, for all $n \geq 1$, with $x_0 = a$ and $x_1 = b$.
(a) Show that $|x_{n+1} - x_n| = 2^{-n}|b - a|$ for all n.

(b) Prove that $\{x_n\}$ converges.

(c) Find $\lim\limits_{n\to\infty} x_n$. *Hint*: Notice that $2x_{n+1} + x_n = 2x_n + x_{n-1}$.

2. Consider a function $f : I \to \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$.
(a) Give a precise definition of the statement "f is uniformly continuous on I".

(b) Suppose f is differentiable on I and f' is bounded on I, prove that f is uniformly continuous on I.

(c) Suppose $f' \neq 0$ on I, prove that either $f'(x) > 0$ or $f'(x) < 0$ for all $x \in I$.

3. (a) Let f be a continuous function on $[0, 1]$. Show that $\lim\limits_{n\to\infty} \int_0^1 x^n f(x)\, dx = 0$.

(b) Given a continuous function h on $[a, b]$, define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = h(x)$, $x \in [a, b]$, $f(x) = h(a)$, $x < a$ and $f(x) = h(b)$, $x > b$.
(i) Show that f is continuous.
(ii) If $f_n(x) := \dfrac{n}{2} \displaystyle\int_{x-1/n}^{x+1/n} f(t)\, dt$, prove that $f_n \to h$ uniformly on $[a, b]$.

4. Any two metrics d_0 and d on a nonempty set X are *equivalent* metrics on X if there exist constants $a, b > 0$ such that $ad_0(x, y) \leq d(x, y) \leq bd_0(x, y)$ for all $x, y \in X$.
(a) Let d_0 and d be equivalent metrics on X. Prove that (X, d_0) is a complete metric space if and only if (X, d) is a complete metric space.

(b) Let $d_0(x, y) = |x - y|$ be the Euclidean metric on $\mathbb{R}$. Let $X = [-1, 1]$. Define another metric by $d(x, y) = |\tan^{-1} x - \tan^{-1} y|$, for all $x, y \in X$. Prove that d_0 and d are equivalent metrics on X. *Hint*: $(\tan^{-1} t)' = \frac{1}{1+t^2}$.

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5. Let $X = C([0,1], \mathbb{R})$ with the metric $d(f,g) = \max\limits_{0 \le x \le 1} |f(x) - g(x)|$. Define the mapping $T : X \to X$ by $T(f)(x) = \frac{1}{3}x^3 + \int\limits_0^x t^2 f(t)dt$ for all $x \in [0,1]$.

(a) Prove that the mapping T has a unique fixed point.

(b) Find the fixed point f of the mapping T. Justify your assertion.

(c) Let $g_0 = 0$. Define $g_{n+1} = T(g_n)$ for all $n \ge 0$. Compute g_1, g_2, and g_3. Verify that g_n is the partial sum of a certain power series and compute the radius of convergence of this power series.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable satisying $f(\mathbf{x}) \ge 0$ for all $\mathbf{x} \in \mathbb{R}^n$, and $f(\mathbf{x}) = 0$ whenever $\|\mathbf{x}\| = 1$.

(a) Prove there is a point $\mathbf{p}$ with $\|\mathbf{p}\| < 1$ such that $f(\mathbf{p}) \ge f(\mathbf{x})$ for all $\mathbf{x}$ with $\|\mathbf{x}\| \le 1$.

(b) For a fixed point $\mathbf{x}$ with $\|\mathbf{x}\| = 1$, prove there exists $t \in (0,1)$ such that $\langle \nabla f(t\mathbf{x}), \mathbf{x} \rangle \le 0$.

(c) Let $n = 2$ and $f(0,0) = 1$. Show that there *cannot* exist $\epsilon_0 > 0$ such that ALL of the following hold.

$$(i)\ f(s,0) \ge 1 + s^2,\ |s| < \epsilon_0 \quad (ii)\ f(0,t) \ge 1 + t^2,\ |t| < \epsilon_0 \quad (iii)\ f(t,t) = 1 + t,\ |t| < \epsilon_0\,.$$

Hint: first establish the value of $\nabla f(0,0)$ if (i)–(iii) were to hold. Justify your assertion.