PhD Comprehensive Exam – Real and Functional Analysis (August 2024)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

Part I. Real Analysis

1. (a) Suppose $\mathcal{S}$ is the smallest σ–algebra on $\mathbb{R}$ containing $\{(r, k] : r \in \mathbb{Q}, k \in \mathbb{Z}\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbb{R}$.

(b) Suppose $\mathcal{S}$ is a σ–algebra on a set X and $A \subseteq X$. Define

$$\mathcal{S}_A = \{E \in \mathcal{S} : A \subseteq E \text{ or } A \cap E = \emptyset\}.$$

(i) Prove that $\mathcal{S}_A$ is a σ–algebra.
(ii) Suppose that $f : X \to \mathbb{R}$ is a function. Prove that f is measurable with respect to $\mathcal{S}_A$ if and only if f is measurable with respect to $\mathcal{S}$ and f is constant on A.

2. (a) Suppose $(X, \mathcal{S}, \mu)$ is a measure space such that $\mu(X) < \infty$ and suppose $f : X \to [0, \infty)$ is an $\mathcal{S}$–measurable function.
(i) Suppose p, r are positive numbers with $p < r$. Prove that if $\int f^r \, d\mu < \infty$, then $\int f^p \, d\mu < \infty$.
(ii) Prove that $\lim\limits_{n\to\infty} \int f^n/(1 + f^n) \, d\mu$ exists and find the limit.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function with $\int |f| \, d\mu < \infty$ for Lebesgue measure λ, that is $f \in \mathcal{L}^1(\mathbb{R})$. Use approximation in $\mathcal{L}^1(\mathbb{R})$ to prove that $\lim\limits_{t\to 0} \int |f(x+t)-f(x)| \, d\lambda(x) = 0$.

3. (a) State the monotone convergence theorem.

(b) Recall that the Maclaurin series for $-\ln(1 - x)$ is

$$-\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots.$$

Apply the monotone convergence theorem to the Maclaurin series for $-\ln(1 - x)$ to show that

$$\int_0^1 -\ln(1 - x) \, dx = \sum_{n=1}^{\infty} \frac{1}{n(n + 1)}$$

and sum the series to find its value. Show all steps and explain where you are applying the montone convergence theorem.

(c) Compute

$$\int_S -\frac{\ln(1 - xy)}{xy} \, d\mu$$

using the methods as in parts (a) & (b), where $S = [0, 1]^2 \subset \mathbb{R}^2$ is the unit square and μ is the planar Lebesgue measure on $\mathbb{R}^2$.

Part II. Functional Analysis

1. For a complex Hilbert space X define the unit ball to be $B_X := \{x \in X : \|x\| \leq 1\}$. Prove directly that a complex Hilbert space X is finite dimensional if and only if B_X is compact. (Note: Here compact means with respect to the topology coming from the norm on X.)

2. (a) Let $C[0,2]$ be the space of real valued continuous functions on $[0,2]$ with norm

$$\|x\| = \max_{0 \leq t \leq 2} |x(t)| .$$

Define the linear functional $f : C[0,2] \to \mathbb{R}$ by $f(x) = \int_0^1 x(t)\, dt - \int_1^2 x(t)\, dt$. Show that f is bounded. Find the norm of f, and prove your answer.

(b) Let X be the space of real valued continuous functions on $[0,2]$ with norm $\|x\| = \int_0^2 |x(t)|\, dt$. Show that X is *not* a Banach space.

(c) Let X be a Banach space and $T : X \to X$ a bounded linear operator on X. Assume $\|T\| < 1$. Denote by T^k the k–th iterate of T: $T^2 x = T(Tx)$, $T^3 x = T(T^2 x)$, ... Show that $\sum_{k=0}^{\infty} T^k x$, converges for each $x \in X$.

3. Let λ be a Lebesgue measure on $[0,1]$. Also let $L^2 := L^2([0,1], \lambda)$ denote the L^2-space of square integrable complex-valued function on $[0,1]$ with respect to the measure λ. Recall that in L^2 we identify functions that agree except on sets of measure zero, and L^2 then consists of (equivalence classes) of functions $f : [0,1] \to \mathbb{C}$ such that $\int_{[0,1]} |f(x)|^2\, d\lambda(x) < \infty$.

We make L^2 into a Hilbert space with the inner product and the corresponding L^2-norm defined respectively, by

$$\langle f, g \rangle := \int_{[0,1]} f(x)\overline{g(x)}\, d\lambda(x) \qquad \|f\|_2 := \left(\int_{[0,1]} |f(x)|^2\, d\lambda(x) \right)^{1/2} .$$

We also define $C([0,1]) := \{f : [0,1] \to \mathbb{C} : f \text{ is continuous}\}$ with the norm

$$\|f\|_{\sup} := \max_{0 \leq x \leq 1} |f(x)| .$$

For each $f \in C([0,1])$ we define the *multiplication operator* $M_f : L^2 \to L^2$ by $M_f(g) := fg$.

(a) Prove that if $f \in C([0,1])$ and $g \in L^2$, then $fg \in L^2$ as claimed. Also prove that $M_f : L^2 \to L^2$ is linear.

(b) Prove that the multiplication operator $M_f : L^2 \to L^2$ is bounded, and that $\|M_f\| = \|f\|_{\sup}$.

(c) Prove that $M_f^* = M_{\overline{f}}$, where M_f^* is the adjoint operator of M_f and $\overline{f} : [0,1] \to \mathbb{C}$ is the complex conjugate of f; i.e., $\overline{f}(x) := \overline{f(x)}$ for all $x \in [0,1]$.