## Comprehensive Exam - Analysis (June 2023)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. (a) A sequence $\left\{x_{n}\right\}$ is called contractive if it satisfies $\left|x_{n+2}-x_{n+1}\right| \leq C\left|x_{n+1}-x_{n}\right|, n \in \mathbb{N}$ for some constant $C, 0<C<1$. Prove that a contractive sequence is convergent.
(b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be differentiable such that $\left|f^{\prime}(x)\right| \leq \frac{1}{2}$ for all $x$. Define $x_{1}=a$ and $x_{n+1}=f\left(x_{n}\right)$, for all $n \in \mathbb{N}$. Prove that the sequence $\left\{x_{n}\right\}$ converges to a limit that is independent of the initial value $x_{1}=a$.
2. Consider the sequence $\left\{f_{n}(x)\right\}$ of partial sums $f_{n}(x)=\sum_{k=1}^{n} \frac{1}{2^{k}} x^{2^{k}-1}$.
(a) Prove that the sequence $\left\{f_{n}(x)\right\}$ converges uniformly to a function $f(x)$ on $[0,1]$.
(b) Show that $f(x)$ satisfies the functional equation $f(x)=\frac{x}{2}+\frac{x}{2} f\left(x^{2}\right)$. Use this relation to evaluate $\int_{0}^{1} f(x) d x$ explicitly.
(c) Prove that the sequence of derivatives $\left\{f_{n}^{\prime}(x)\right\}$ converges pointwise but not uniformly on the interval $(-1,1)$.
3. Let $(X, d)$ be a metric space. A mapping $F: X \rightarrow \mathbb{R}$ is called Lipschitz if there is a constant $C>0$ such that $|F(p)-F(q)| \leq C d(p, q)$ for all $p, q \in X$.
(a) If $\left\{p_{k}\right\}$ is a Cauchy sequence in $X$, and if $F: X \rightarrow \mathbb{R}$ is Lipschitz, then prove that $\left\{F\left(p_{k}\right)\right\}$ is convergent.
(b) Let $X=C([0,1])$ with the metric $d(f, g)=\max _{0 \leq x \leq 1}|f(x)-g(x)|$. Prove that the mapping $F: X \rightarrow \mathbb{R}$ defined by $F(f)=\max _{0 \leq x \leq 1} f(x)$ is Lipschitz.
(c) Use parts (a) and (b) to prove that if a sequence of continuous functions $\left\{f_{n}:[0,1] \rightarrow \mathbb{R}\right\}$ converges uniformly to a function $f:[0,1] \rightarrow \mathbb{R}$, then the sequence $\left\{m_{n}=\max _{0 \leq x \leq 1} f_{n}(x)\right\}$ converges to $m=\max _{0 \leq x \leq 1} f(x)$.
4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Define $F(x)=\int_{0}^{x}(x-t) f(t) d t$ for all $x$.
(a) Prove that $F^{\prime \prime}(x)=f(x)$.
(b) Use part (a) to prove that for each $x>0$, there is $t_{0} \in(0, x)$ such that $F(x)=\frac{1}{2} f\left(t_{0}\right) x^{2}$.
5. (a) Prove that the set $X$ of all integers with metric $d(m, n)=|m-n|, m, n \in \mathbb{Z}$ is a complete metric space.
(b) Let $X$ be the set of all positive integers with metric $d(m, n)=\left|\frac{1}{m}-\frac{1}{n}\right|$. Prove that $(X, d)$ is not complete.
(c) Let $(X, d)$ be a sequentially compact metric space and let $Y \subset X$ such that $(Y, d)$ is complete. Prove that $(Y, d)$ is sequentially compact.
6. Let $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be continuously differentiable.
(a) Suppose that $f(1,0)=f(-1,0)$. Consider the restriction of $f$ on the unit circle given by $\psi(t)=f(\cos t, \sin t), 0 \leq t<2 \pi$. Prove there exist two distinct points $(x, y)$ on the unit circle such that $-y \frac{\partial f}{\partial x}(x, y)+x \frac{\partial f}{\partial y}(x, y)=0$
(b) Next suppose $f(0,0)=0$. Fix a point $(u, v) \neq(0,0)$. Prove that

$$
f(u, v)=\int_{0}^{1}\left[u \frac{\partial f}{\partial x}(t u, t v)+v \frac{\partial f}{\partial y}(t u, t v)\right] d t .
$$

Verify this result when for example, $\nabla f(x, y)=(x, y)$.

