## Comprehensive Exam - Analysis (Jan 2023)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let $f:[0,1] \rightarrow \mathbb{R}$ satisfy $x<f(x)<1$ for all $0 \leq x<1$, and $f(1)=1$. Define $x_{1}=f(0)$ and $x_{n+1}=f\left(x_{n}\right)$, for all $n=1,2,3, \ldots$.
(a) Prove that the sequence $\left\{x_{n}\right\}$ is well defined and convergent.
(b) Prove that if $f:[0,1] \rightarrow \mathbb{R}$ is continuous, then $\lim x_{n}=1$. (Hint: Suppose not.)
2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be bounded and twice differentiable.
(a) Suppose that $f^{\prime \prime}(x) \geq 0$ for all $x \geq 0$. Prove that

$$
\text { (i) } \lim _{x \rightarrow \infty} f^{\prime}(x)=0 \quad \text { (ii) } \lim _{x \rightarrow \infty} f(x) \quad \text { exists }
$$

(b) If instead $f^{\prime \prime}(x) \geq 0$ for all $x \in \mathbb{R}$, prove that $f$ is constant.
3. Define a sequence of functions by $f_{n}(x)=\sum_{k=1}^{n} \frac{x^{2 k}}{k+x^{2 k}}, n=1,2,3, \ldots$.
(a) Prove that $\left\{f_{n}\right\}$ converges pointwise on $[0,1)$.
(b) Prove that $\left\{f_{n}\right\}$ does not converge uniformly on $[0,1)$.
4. (a) Prove that a closed subspace of a complete metric space is also complete.
(b) Let $A=\left\{f \in C[0,1]: \int_{0}^{1} f(x) \sin (\pi x) d x=0\right\}$ be a subspace of the metric space of continuous functions on $[0,1]$ with metric $d(f, g)=\sup _{x \in[0,1]}|f(x)-g(x)|$. Prove that $A$ is complete. (Hint: Use the result in part (a)).
5. Let $(X, d)$ be a metric space.
(a) Let $\left\{x_{n}\right\}$ be a Cauchy sequence in $X$. Prove that $f(x)=\lim _{n \rightarrow \infty} d\left(x, x_{n}\right), x \in X$ defines a function $f: X \rightarrow \mathbb{R}$, and that $f(x)$ is continuous.
(b) Suppose that every continuous function $f: X \rightarrow \mathbb{R}$ attains a minimum value. Then prove that $(X, d)$ is complete. (Hint: Use part (a)).
6. Consider the function on $\mathbb{R}^{2}$ defined by

$$
f(x, y)=\left(x^{2}+y^{2}\right) \sin \frac{1}{\sqrt{x^{2}+y^{2}}}, \quad(x, y) \neq(0,0), \quad f(0,0)=0
$$

where $f_{x}, f_{y}$ denote the first partial derivatives of $f$.
(a) Show that $f$ satisfies the first order approximation at $(0,0)$

$$
\lim _{(x, y) \rightarrow(0,0)} \frac{f(x, y)-f(0,0)-x f_{x}(0,0)-y f_{y}(0,0)}{\sqrt{x^{2}+y^{2}}}=0 .
$$

(b) A sufficient condition for part (a) to hold is that the first partial derivatives $f_{x}, f_{y}$ are continuous in an open neighborhood in $\mathbb{R}^{2}$ containing $(0,0)$. Prove that this condition is however not necessary by showing that $f_{x}, f_{y}$ are not continuous at $(0,0)$.

