Comprehensive Exam – Analysis (January 2016)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, with $f(0) = 0$ and $|f'(0)| < 1$.

(a) Prove there exists $r > 0$ so that $M := \max\limits_{x \in [-r,r]} |f'(x)| < 1$.

(b) Let $I = [-r, r]$ for $r > 0$ given by part (a). Define a sequence $\{x_n\}$ by

$$x_{n+1} = f(x_n), \quad n \geq 0, \quad x_0 = r.$$

Show that (i) $x_n \in I$ for all $n \geq 0$, and (ii) $\lim\limits_{n \to \infty} x_n = 0$.

2. (a) Prove that $f(x) = \sum\limits_{n=1}^{\infty} n x^n$ converges uniformly on $[-r, r]$ for any $0 < r < 1$.

(b) Evaluate $f(\frac{1}{2})$ and $f'(\frac{1}{2})$ as explicit rational numbers.

(c) Prove that the series in part(a) does not converge uniformly on $(-1, 1)$.

3. Let $f(x)$ be a continuous function on $[a, b]$, $a < b$.

(a) Show that there exists $c \in [a, b]$ such that $\dfrac{1}{b - a} \displaystyle\int_a^b f(x)\, dx = f(c)$.

(b) If $\displaystyle\int_a^b f(x)^2\, dx = 0$, show that $f(x) = 0$ on $[a, b]$.

4. Let $X = (X, d)$ be a metric space.

(a) Give the definition of a contraction mapping $T : X \to X$.

(b) Let $X = C([0, 1])$ be the metric space of all real valued continuous functions on $[0, 1]$, with the metric $d(f, g) := \sup\limits_{x \in [0,1]} |f(x) - g(x)|$. Prove that $T : X \to X$ defined by

$$T(f)(x) := \frac{1}{2} \int_0^x \left(f(t) - e^t \right) dt, \quad 0 \leq x \leq 1,$$

is a contraction mapping.

(c) Verify that $f(x) = e^{x/2} - e^x$ is a fixed point of T defined in part (b), and show that the fixed point is unique.

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5. (a) Give a precise definition of a sequentially compact metric space.

(b) Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$ be a continuous function. If X is sequentially compact then prove that the image $f(X) \subset Y$ is also sequentially compact with respect to the metric d'.

(c) Let (X, d) be a sequentially compact metric space. Suppose $f : X \to \mathbb{R}$ is a continuous function with the property: for each $x \in X$, there exists $x' \in X$ such that $|f(x')| \leq \frac{1}{2}|f(x)|$. Prove that there exists a point $x_0 \in X$ such that $f(x_0) = 0$.

6. Suppose a real valued function $f(x, y)$ is defined on an open set $U \in \mathbb{R}^2$. Assume that the first partial derivatives of $f(x, y)$ exist and are uniformly bounded on U.

(a) Prove that $f(x, y)$ is continuous on U.

(b) Prove that part (a) does not hold without the boundedness of the first partial derivatives using the following example:

Let $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show explicitly that the first partial derivatives of $f(x, y)$ exist but are *not* bounded in an open neighborhood of $(0, 0)$. Then show that $f(x, y)$ is not continuous at $(0, 0)$.