PhD Comprehensive Exam – Ring Theory (August 2025)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page, and start each problem on a new page.

Throughout, R denotes an associative ring with identity.

1. (Modules) Let M be a left R-module.
   (a) Prove that if M is noetherian, then every R-submodule of M is finitely generated.
   (b) Let N be a finitely generated R-submodule of M. Prove that M is finitely generated as a left R-module if and only if M/N is.
   (c) Give an example to show that the statement in (b) does not hold if N is not finitely generated.

2. (Semisimple Rings)
   (a) Prove that for any positive integer n, a matrix belongs to the center of $\mathbb{M}_n(R)$ if and only if it is of the form rI_n, where r is in the center $Z(R)$ of R, and I_n is the identity matrix.
   (b) Prove that if R is a simple ring, then $Z(R)$ a field. (You may assume that the center of a ring is a ring.)
   (c) Prove that if R is a semisimple ring, then $Z(R)$ is a finite direct product of fields.

3. (Semiprime Rings and Ideals)
   (a) Give an example of ring that is semiprime but not semisimple. No justification required.
   (b) Prove that R is semiprime if and only if the polynomial ring $R[x]$ is semiprime.
   (c) Prove that every ideal of R is semiprime if and only if every ideal of R is idempotent (i.e., $I^2 = I$ for every ideal I).
   (d) Prove that if R is commutative, then R is von Neumann regular (i.e., for all $r \in R$ there exists $p \in R$ such that $r = rpr$) if and only if every ideal of R is semiprime.

4. (Leavitt Path Algebras and Related Ideas) Let E be a finite graph. Suppose that c is a cycle in E with $s(c) = v$, and that the edge e is an exit for c with $s(e) = v$.
   (a) i. Prove that $L_K(E)v = L_K(E)cc^* \oplus L_K(E)(v - cc^*)$ as left ideals of $L_K(E)$.
   ii. Prove that $L_K(E)v \cong L_K(E)cc^*$.
   iii. Prove that $L_K(E)(v - cc^*)$ is nonzero.
   (b) Prove that $L_K(E)v$ does not have the descending chain condition.

5. (Primitive Rings)

   (a) Prove that if R is left primitive, then it is prime.

   (b) Let K be a field, and V a (right) K-vector space with basis $B = \{e_i \mid i \in I\}$, for some indexing set I. For all $i, j \in I$ let $E_{ij} \in \mathrm{End}_K(V)$ denote the "matrix unit" defined by $E_{ij}(e_j) = e_i$ and $E_{ij}(e_k) = 0$, for all $k \in I \setminus \{j\}$. Show that any K-subalgebra S of $\mathrm{End}_K(V)$ that contains $\{E_{ij} \mid i, j \in I\}$ is left primitive.

   (c) Supposing that R is left primitive, must $R[x]$ also be left primitive?

6. (Local and Semilocal Rings)

   (a) Prove that if $R \neq 0$ and every non-unit of R is nilpotent, then R is local.

   (b) Suppose that there exists a non-unit $r \in R$ such that every $p \in R \setminus \{0\}$ can be expressed as $p = r^n u$, for some $n \in \mathbb{N}$ and unit $u \in R$. Prove that R is local.

   (c) Prove that if R is semilocal, then for any ideal I of R the natural map $U(R) \to U(R/I)$ is onto. (Here $U(R)$, respectively $U(R/I)$, denotes the set of units of R, respectively R/I.)