PhD Comprehensive Exam – Ring Theory (January 2024)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Please write only on one side of the page and start each problem on a new page.

Throughout, R denotes an associative ring with identity. Homomorphisms of left modules will be written on the right: so we write $(m)f$, and fg means 'first f, then g'.

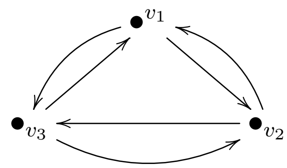
1. (Modules) Let M be a left R-module.
   (a) Show that if M is noetherian, then every R-submodule of M is finitely generated.
   (b) Let N be a finitely generated R-submodule of M. Prove that M is finitely generated as a left R-module if and only if M/N is.
   (c) Give an example to show that the statement in (b) does not hold if N is not finitely generated.

2. (Semisimple Rings)
   (a) Show that for any positive integer n, a matrix belongs to the center of $\mathbb{M}_n(R)$ if and only if it is of the form rI_n, where r is in the center $Z(R)$ of R, and I_n is the identity matrix.
   (b) Show that if R is a simple ring, then $Z(R)$ a field. (You may assume that the center of a ring is a ring.)
   (c) Show that if R is a semisimple ring, then $Z(R)$ is a finite direct product of fields.

3. (Semiprime Rings and Ideals)
   (a) Give an example of ring that is semiprime but not semisimple. No justification required.
   (b) Show that R is semiprime if and only if the polynomial ring $R[x]$ is semiprime.
   (c) Show that every ideal of R is semiprime if and only if every ideal of R is idempotent (i.e., $I^2 = I$ for every ideal I).
   (d) Show that if R is commutative, then R is von Neumann regular (i.e., for all $r \in R$ there exists $p \in R$ such that $r = rpr$) if and only if every ideal of R is semiprime.

4. (Functors and Natural Transformations) Let $S = \mathbb{M}_2(R)$.
   (a) Explicitly define an equivalence functor $F : S\text{Mod} \to R\text{Mod}$. Prove that your functor really is an equivalence of categories, by giving a functor $G : R\text{Mod} \to S\text{Mod}$, and verifying that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors on the appropriate categories.
   (b) Referring to (a), is it the case that ${}_RF({}_SS) \cong {}_RR$ in RMod? Discuss.

5. (Leavitt Path Algebras and Related Ideas) Throughout this question, E denotes the graph



   (a) Compute 'directly' the graph monoid M_E of E. (First find a set of representatives of the equivalence classes in M_E, then prove that these equivalence classes are distinct.)

   (b) The theorem of Ara / Moreno / Pardo gives that $M_E \cong \mathcal{V}(L_K(E))$. Under this identification, which element of M_E corresponds to $[1_{L_K(E)}]$?

   (c) Give the matrix $I - A_E^t$, and compute its Smith normal form. Explain why this Smith normal form is consistent with your answer to (a).

   (d) True or False: $L_K(E) \cong L_K(1,5)$. Fully justify. (Here $S = L_K(1,5)$ denotes the Leavitt algebra for which ${}_SS^1 \cong {}_SS^5$ as left S-modules.)

6. (Leavitt Path Algebras and Related Ideas) Let E be a finite graph. Suppose that c is a cycle in E with $s(c) = v$, and that the edge e is an exit for c with $s(e) = v$.

   (a) i. Prove that $L_K(E)v = L_K(E)cc^* \oplus L_K(E)(v - cc^*)$ as left ideals of $L_K(E)$.

   ii. Prove that $L_K(E)v \cong L_K(E)cc^*$.

   iii. Prove that $L_K(E)(v - cc^*)$ is nonzero.

   (b) Prove that $L_K(E)v$ does not have the descending chain condition.