PhD Preliminary Exam – Linear Algebra (January 2026)

Attempt ANY 5 of the following 6 problems. CROSS OUT any problem that you do not want to be graded. Each problem is worth 20 points. Please write only on one side of the page and start each problem on a new page.

1. Let V be a vector space over a field F and suppose that W_1 and W_2 are subspaces of V.

(a) Prove that $W_1 \cap W_2$ is a subspace of V.

(b) Let $W_1 + W_2$ be defined by the set of all vectors in V of the form $v = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$. Prove $W_1 + W_2$ is a subspace of V.

(c) Suppose V is finite dimensional. Prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)\,.$$

2. (a) Let V and W be finite dimensional vector spaces over a field F, and $T : V \to W$ be linear. Let β and γ be ordered bases for V and W, respectively, and $A = [T]_\beta^\gamma$. Prove that $\dim(\mathrm{N(T)}) = \dim(\mathrm{N(L_A)})$.

(b) Let $\mathcal{P}_n$ denote the vector space of polynomials in x of degree less than or equal to n with real coefficients and $T : \mathcal{P}_3 \to \mathcal{P}_1$ be a linear transformation defined by $T(p)(x) = \frac{d^2p}{dx^2}$.

(i) Verify part (a) using ordered bases $\beta = \{1, x, x^2, x^3\}\,, \quad \gamma = \{1, x\}$.

(ii) Prove that $N(T) = R(T)$.

3. Let A and B be $m \times n$ and $n \times p$ matrices over $\mathbb{R}$, respectively.

(a) Prove that $\dim(N(AB)) \leq \dim(N(A)) + \dim(N(B))$.
[Hint: Consider $V = \{x \in \mathbb{R}^p : ABx = 0\}$ and $W = \{y = Bx : x \in \mathbb{R}^p, Ay = 0\}$. Then consider the map $L : V \to W$ defined by $L(x) = Bx$ for all $x \in V$.]

(b) Prove that

$$\mathrm{rank}(A) + \mathrm{rank}(B) - n \leq \mathrm{rank(AB)} \leq \min\{\mathrm{rank(A)}, \mathrm{rank(B)}\}.$$

4. Two matrices A and B are said to be similar if $B = SAS^{-1}$, for some invertible matrix S.

(a) Show that A and B have the same determinant.

(b) Show that A and B share the same eigenvalues and corresponding algebraic multiplicity. From this, show that A and B have the same trace.

(c) Determine the relationship between the eigenvectors of A and B.

(d) Show that the matrices $B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ are similar by finding the corresponding matrix S and its inverse.

5. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and W a nontrivial subspace of V. Consider $P : V \to V$ be the orthogonal projection onto W, defined as

$$P(v) = w, \text{ where } v = w + w', \text{ with } w \in W \text{ and } w' \in W^{\perp}, \text{ the orthogonal complement of } W.$$

(a) Show that $P^2 = P$.

(b) Show that for every $v \in V$, $\langle P(v), v \rangle \geq 0$.

(c) Show that for every $v \in V$, $\|P(v)\| \leq \|v\|$.

(d) Show that $\|v\|^2 = \|P(v)\|^2 + \|(I - P)v\|^2$.

6. Consider the real vector space $V = \mathcal{P}_3(\mathbb{R})$ of all polynomials with real coefficients of degree at most 3. Define the operator

$$T(p(x)) = xp''(x) - p'(x) + 2p(x), \quad p(x) \in \mathcal{P}_3(\mathbb{R}).$$

(a) Show that T is a linear transformation on V and find $[T]_\beta$, the matrix representation of T in the standard basis $\beta = \{1, x, x^2, x^3\}$ for V.

(b) Determine the characteristic polynomial *and* the minimal polynomial of T.

(c) Determine the Jordan form of T.